CHARACTERISTIC CLASSES AND THE CHERN CHARACTER, JAMIE WALTON,

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1. Introduction

1.1. **What is a characteristic class?** A characteristic class is a way of assigning cohomology classes to vector bundles in a natural way. They can detect whether a locally trivial bundle is globally twisted.

1.2. **What use are they?**

1.2.1. *Distinguishing bundles.* Just as cohomology groups can tell apart spaces, characteristic classes can help to distinguish vector bundles. In particular, sometimes (but not always) a characteristic class can detect that a vector bundle is non-trivial ('essentially twisted'). Triviality of a vector bundle is not always obvious; for example, it was a significant achievement in algebraic topology[1] to prove that the only parallelisable spheres (those with trivial tangent bundles) are S^0, S^1, S^3 and S^7. This implies that the only finite-dimensional division algebras have dimensions 1, 2, 4 and 8. These problems were actually solved using higher cohomology operations rather than characteristic classes directly, but characteristic classes form an integral part of the structure.

1.2.2. *Link between cohomology and K-theory: the Chern character.* K-theory is a natural invariant for C^*-algebras, so is probably of more interest to many of us in the analysis group than singular cohomology of spaces. However, for a given space its cohomology is often easier to compute than its K-theory (for example, we can compute it given a nice cell decomposition) and potentially provides different geometric information. The two are linked via the *Chern character*. We'll see how one defines the Chern character via the Chern characteristic classes. The Chern character plays an important role in topological index theorems for elliptic operators.

1.2.3. *Other miscellaneous uses.* Apparently characteristic classes appear in mathematical physics and other areas. From wikipedia: "*[The Chern classes] have since found applications in physics, Calabi–Yau manifolds, string theory, Chern–Simons theory, knot theory, Gromov–Witten invariants, topological quantum field theory, the Chern theorem etc.*"

Characteristic classes solve the cobordism problem. Two closed, smooth n-manifolds M and N are unoriented cobordant (that is, there is an $(n+1)$-manifold W with boundary $M \sqcup N$) if and only if the Stiefel–Whitney numbers of the manifolds agree (these are integers constructed from the Stiefel–Whitney classes of the tangent bundles of the two manifolds). Manifolds are oriented cobordant if and only if both their Stiefel–Whitney and Pontryagin numbers

[1]Actually a lot more can be said. Adams found a precise formula for the maximum number of linearly independent vector fields on the spheres.

agree. There is sometimes hope of computing these classes for manifolds; for example, Chern–Weil Theory sets out an approach via connections and curvature forms on the manifolds involved.

Bounds on dimensions in which manifolds can be immersed can sometimes be given using characteristic classes. For example, if an n-manifold can be immersed into $\mathbb{R}^{n+k}$, then the inverse of the total Stiefel–Whitney class of the tangent bundle is zero in degrees above k. One can prove, for example, that 2^ℓ-projective space ($\ell \geq 1$) can be immersed into $\mathbb{R}^{2^\ell+k}$ if and only if $k \geq 2^\ell - 1$.

2. Vector bundles

A vector bundle over a space B (the 'base space') is given by 'continuously attaching vector spaces to each point of B'.

Definition 2.1. *A (real)* **vector bundle** ν *consists of:*

- *a space $B = B(\nu)$ called the* **base space***;*
- *a space $E = E(\nu)$ called the* **total space***;*
- *a map $\pi = \pi(\nu)\colon E \to B$ called the* **projection***;*
- *a finite-dimensional $\mathbb{R}$-vector space structure on each* **fibre** $\pi^{-1}(b)$.

We require this data to be 'locally trivial', which means the following:

For each $b \in B$ there exists a neighbourhood $U \subseteq B$ of b, an integer $r \geq 0$ and a commutative diagram

$$\begin{tikzcd} U \times \mathbb{R}^r \arrow[rr, "h"] \arrow[dr, "\pi_1"'] & & \pi^{-1}(U) \arrow[dl, "\pi"] \\ & B & \end{tikzcd} \tag{2.1}$$

where h is a homeomorphism, $\pi_1(u,x) := u$ and h restricts to isomorphisms on the fibres. (Note that the above diagram commuting is equivalent to specifying that h maps fibres to fibres.)

The dimension r of the fibre must be locally constant; usually this number is globally constant, in which case we call r the **rank** of the bundle.

We have the analogous notion of a complex vector bundle, where we instead attach $\mathbb{C}$-vector spaces to each point of B. Every $\mathbb{C}$-vector bundle of rank r defines an $\mathbb{R}$-vector bundle of rank $2r$, as one would imagine. Conversely, given an even dimensional $\mathbb{R}$-vector bundle ν, one can define a corresponding $\mathbb{C}$-vector bundle if ν is equipped with a 'complex structure', a self-map J of the total space taking fibres linearly to themselves and on such a fibre satisfying $J(J(v)) = -v$.

Definition 2.2. *Two vector bundles ν and ξ over B are called* **isomorphic** *if there exists a homeomorphism $h\colon E(\nu) \to E(\xi)$ which maps fibres isomorphically to fibres.*

Let us write $\mathrm{Vect}_r^{\mathbb{R}}(X)$ for the set of isomorphism classes of real rank r bundles over X (and similarly for complex bundles), just $\mathrm{Vect}_r(X)$ when the field is understood and dropping the r if we include bundles of all ranks.

One can also talk of maps between vector bundles over different base spaces:

Definition 2.3. *A* **bundle map** *between bundles ν and ξ is a commutative diagram of maps*

$$\begin{array}{ccc} E(\nu) & \xrightarrow{g} & E(\xi) \\ {\scriptstyle \pi(\nu)}\downarrow & & \downarrow{\scriptstyle \pi(\xi)} \\ B(\nu) & \xrightarrow{f} & B(\xi) \end{array}$$

for which g is linear on each fibre.

Given a map $f\colon B_1 \to B_2$ and a vector bundle ν over B_2, with projection map $\pi\colon E \to B_2$, one can pull it back to a bundle $f^*\nu$ over B_1 (and functorially, so $(f \circ g)^*\nu = g^*(f^*\nu)$). The **pullback bundle** $f^*\nu$ has total space

$$E(f^*\nu) := \{(b, e) \in B_1 \times E \mid f(b) = \pi(e)\}$$

with the subspace topology. The projection map is the projection to the first coordinate. This defines a bundle map from $f^*\nu$ to ν. Conversely, a bundle map taking fibres isomorphically to fibres is essentially a pullback:

Lemma 2.4. *Given a bundle map between ξ and ν mapping fibres isomorphically to fibres, with induced map f between the base spaces, the bundle ξ is isomorphic to the pullback $f^*\nu$.*

Let's now look at the basic examples of vector bundles:

Example 2.5. Given any space B we have the **trivial bundle** with total space $B \times \mathbb{R}^n$ and projection map given by the projection to the first coordinate. A bundle is called trivial if it is isomorphic to the trivial bundle. Equation 2.1 specifies that any bundle is trivial when restricted to sufficiently small neighbourhoods; a trivial bundle is one that is trivial globally rather than just locally.

A section of a bundle ν with projection $\pi\colon E \to B$ is a map $s\colon B \to E$ for which $\pi \circ s = \mathrm{id}_B$ (that is, s maps a point of b to a point in its fibre). A rank r bundle is trivial if and only if it has r sections which are linearly independent on each fibre.

Example 2.6. Given a smooth n-manifold M, we have its associated rank n 'tangent bundle' τ_M. The projection map of τ_M sends a tangent vector to the point of M at which it is based.

A section of the tangent bundle is called a vector field; it is a continuous choice of tangent vector at each point of the manifold. The manifold is called 'parallelisable' if τ_M is a trivial bundle. For example, S^1 is parallelisable (there is an obvious non-zero vector field) whereas S^2 isn't since it doesn't even have a single non-zero vector field (which is also the case for all even-dimensional spheres, this is the 'Hairy Ball Theorem').

A smooth map $f\colon M \to N$ induces a bundle map between their tangent bundles (this is functorial, generalising the 'chain rule' $(f \circ g)' = (f' \circ g) \cdot g'$ from Real Analysis). The tangent vectors at $m \in M$ are mapped to tangent vectors at $f(m)$ by the Jacobian.

Example 2.7. One may visualise the rank 1 trivial bundle over S^1 as a cylinder. Adding a twist gives the 'Möbius bundle', the simplest non-trivial bundle. This is generalised by the 'tautological line bundle over projective n-space'.

We've seen that given a fibre bundle one may construct a new one by pulling it back with a continuous map. There are lots of other more algebraic constructions for them too; basic operations on vector spaces generally have counterparts for vector bundles.

For example, one may take the *Whitney sum* $\nu \oplus \xi$ of two vector bundles over a common base space. One should think of each fibre of this new bundle as the direct sum of the fibres of the originals. Similarly one has $\nu \otimes \xi$, $\hom(\nu, \xi)$ (and so dual vector bundles, $\hat{\nu} = \hom(\nu, T)$ with T the rank 1 trivial bundle over X),...

2.1. **Classifying spaces.**

We assume throughout that all spaces are Hausdorff and paracompact (this isn't asking much; all compact Hausdorff spaces and all CW complexes satisfy this, for example).

We have the following beautiful result:

Theorem 2.8. *For each $r \in \mathbb{N}$ there is a rank r* **universal bundle** *γ_r for which any rank r bundle is a pullback of γ_r.*

Moreover, writing the base space of γ_r as $B(r)$, two maps f, $g\colon X \to B(r)$ give isomorphic pullbacks $f^*\gamma_n \cong g^*\gamma_n$ if and only if f and g are homotopic. So taking pullbacks of γ_r gives a natural bijection

$$[X, B(r)] \xrightarrow[f \mapsto f^*\gamma_r]{\cong} \mathrm{Vect}_r(X),$$

where $[X, B(r)]$ denotes the set of homotopy classes of maps from X to $B(r)$.

Let's just say a few of words on this space $B(n)$. It can be constructed as the Grassmannian of n-planes in $\mathbb{R}^\infty$ (or $\mathbb{C}^\infty$, as appropriate), or as the inductive limit of compact, finite-dimensional Grassmannians $\mathrm{Gr}(n, \mathbb{R}^k)$ of n-planes in $\mathbb{R}^k$ ($\mathbb{C}^k$, resp.). In the complex case we call this space[2] $BU(n)$. It is the **classifying space** of $\mathbb{C}$-vector bundles.

For $n = 1$ one has $BU(1) \cong \mathbb{C}P^\infty$, the infinite-dimensional complex projective space of rank 1 subspaces of $\mathbb{C}^\infty$. The universal rank 1 bundle over $\mathbb{C}P^\infty$ consists of choices of a point of $\mathbb{C}P^\infty$ (a complex line in $\mathbb{C}^\infty$) and a point of the corresponding complex line. This is sometimes called the rank 1 (complex) **tautological bundle**.

The above shows that the study of vector bundles can be recast as the homotopy theory of the base spaces in question and the classifying spaces of bundles.

Here's a fun application:

Example 2.9. What $\mathbb{C}$-vector bundles do spheres support? Recall that the nth homotopy group $\pi_n(X)$ of a (path-connected) space X has as elements the homotopy classes of maps from S^n to X, so in fact we have bijections

$$\mathrm{Vect}_r^{\mathbb{C}}(S^n) \cong [S^n, BU(r)] \cong \pi_n(BU(r)) \cong \pi_{n-1}(U(r))$$

(the final bijection is a standard fact about classifying spaces of topological groups).

So to work out what $\mathbb{C}$-vector bundles spheres support is to work out the homotopy groups of the unitary groups. For example, since the unitary groups are connected, S^1 only supports

[2]...since it is the classifying space of $U(n)$, for those familiar with classifying spaces of groups. This space classifies principal $U(n)$ bundles. Over paracompact base spaces complex bundles can be give a 'hermitian metric' which then canonically defines a principal $U(n)$ bundle; conversely a principal $U(n)$ bundles defines a rank n vector bundle. Of course we have similar things for real bundles, where the classifying space is $BO(n)$.

trivial $\mathbb{C}$-bundles; S^2 has a $\mathbb{Z}$-worth of rank r-vector bundles for each r (generated by a single line bundle, in some sense). More details and examples in the appendix.

3. Cohomology

We just recall the basics, for notation. For each $n \in \mathbb{N}_0$ and topological space X one has the Abelian group $H^n(X)$, called the **degree n (singular) cohomology group of X**. For the *co*homology groups it is advantageous to view them all together, since there is also a ring structure: we write

$$H^*(X) = \bigoplus_{n=0}^{\infty} H^n(X),$$

which we view, firstly, as a graded Abelian group. There is also a product operation

$$\smile : H^*(X) \times H^*(X) \to H^*(X),$$

where we write $a \smile b$ for the so-called **cup product** of cohomology classes a and b. Everything is associative, distributive and there's a unit $1 \in H^0(X)$.

If a cohomology class x is purely graded in some $H^n(X)$, then we say that x has **degree** n and write $|x| = n$. For purely graded a, $b \in H^*(X)$ we have that $|a \smile b| = |a| + |b|$ (the product respects the grading) and moreover the cohomology ring is **graded commutative**, which means that $a \smile b = (-1)^{|a| \cdot |b|}(b \smile a)$.

Cohomology is a *contravariant functor*: not only given a space X do we get a ring $H^*(X)$, whenever we're given a continuous map f we get a ring homomorphism $f^* \colon H^*(Y) \to H^*(X)$ (the direction of the arrow is reversed because this is a *contravariant* functor). Being a functor means that it respects the structure (identities and composition) of the categories we're mapping to and from: we have that $(\mathrm{id}_X)^* = \mathrm{id}_{H^*(X)}$ and $(f \circ g)^* = g^* \circ f^*$. In fact, it defines a functor from the homotopy category: if f and g are homotopic then $f^* = g^*$.

Loosely, the cohomology group $H^n(X)$ can detect "n-dimensional holes" in your space X:

Example 3.1. The cohomology ring of the 2-sphere S^2 is:

$$H^*(S^2) \cong \mathbb{Z}[x]/(x^2)$$

where $|x| = 2$. That is, a generic element is $k_1 + k_2 x$ where each $k_i \in \mathbb{Z}$, $H^0(S^2) \cong \mathbb{Z}$, $H^1(S^2) \cong 0$, $H^2(S^2) \cong \mathbb{Z}$ (generated by x) and $H^n(S^2) \cong 0$ for $n > 2$. Triviality of $H^1(S^2)$ is related to the fact that any 1-dimensional loop on S^2 can be continuously shrunk to a point.

Example 3.2. The cohomology ring of the 2-torus $\mathbb{T}^2 \coloneqq S^1 \times S^1$ is

$$H^*(\mathbb{T}^2) \cong \mathbb{Z}[x, y]/(x^2, y^2).$$

So a typical element is of the form $k + (mx + ny) + \ell(x \smile y)$ where k, m, n and $\ell \in \mathbb{Z}$; we have $H^0(\mathbb{T}^2) \cong \mathbb{Z}$, $H^1(\mathbb{T}^2) \cong \mathbb{Z}^2$ (generated by x and y), $H^2(\mathbb{T}^2) \cong \mathbb{Z}$ (generated by $x \smile y = -(y \smile x)$) and $H^n(\mathbb{T}^2) \cong 0$ otherwise. That $H^1(\mathbb{T}^2) \cong \mathbb{Z}^2$ corresponds to the fact that there are two distinct and 'primitive' loops on the torus, wrapping around the meridian and the longitude.

Example 3.3. The one-point join of spheres $S^2 \vee S^4$ and the space $\mathbb{C}P^2$ have the same cohomology groups in all degrees (isomorphic to $\mathbb{Z}$ in degrees 0, 2 and 4, and are trivial otherwise). However, they are not homotopy equivalent. Although this isn't detected by $H^*(-)$ as a *graded group*, it is detected by its ring structure: the element $\alpha \smile \alpha$, where α is a generator in degree 2, is a generator in degree 4 in the cohomology ring of $\mathbb{C}P^2$ whereas it is trivial for $S^2 \vee S^4$.

4. The Chern Classes

A characteristic class sends a vector bundle over X to an element of[3] $H^*(X)$. We'll look at the Chern classes. These are characteristic classes for complex vector bundles, so for the rest of this section our vector bundles will be over $\mathbb{C}$.

It turns out that the Chern classes are completely characterised by a useful set of axioms:

Definition 4.1. *The (total)* **Chern class** *of a complex vector bundle ν is an element of $H^*(X)$, denoted by $c(\nu)$. The Chern classes are concentrated in even degrees $H^{2n}(X)$ of the cohomology and we denote the degree $2n$ part of $c(\nu)$ by $c_n(\nu)$.*

The Chern classes are uniquely determined by the following axioms:

1. *$c_0(\nu) = 1$ and $c_k(\nu) = 0$ for $k > r$ for all rank r vector bundles;*
2. *for a continuous map f we have[4] that $c(f^*\nu) = f^*c(\nu)$;*
3. *for two vector bundles ν and ξ over X we have $c(\nu \oplus \xi) = c(\nu) \smile c(\xi)$;*
4. *for the rank 1 tautological bundle γ_1 we have that $c_1(\gamma_1)$ is the canonical degree 2 generator of $H^*(\mathbb{C}P^\infty)$.*

The second axiom may be restated as saying that the Chern classes are *natural*: c defines a natural transformation from the functor $\mathrm{Vect}^{\mathbb{C}}(-)$ to $H^*(-)$ (both regarded as just sets). The third axiom is sometimes called the *Whitney sum formula*; degree-wise, it can be rewritten:

$$c_n(\nu \oplus \xi) = \sum_{i+j=n} c_i(\nu) \smile c_j(\xi).$$

The final axiom may be considered as a *normalisation axiom*. As we'll mention later, $H^*(\mathbb{C}P^\infty) \cong \mathbb{Z}[x]$ where $|x| = 2$.

One can derive some simple properties from the above axioms. For example, the Chern classes of a trivial bundle are trivial:

Proposition 4.2. *If ν is a trivial bundle then $c(\nu) = 1$.*

Proof. One may easily show that a trivial bundle is isomorphic to the pullback $p^*\xi$ of the trivial ξ bundle over the one-point space $*$, where $p\colon X \to *$. Since $H^*(*) = \mathbb{Z}$ has no higher cohomology, by Axiom 1 above we have that $c(\xi) = 1$, and by naturality $c(\nu) = c(p^*\xi) = p^*c(\xi) = p^*(1) = 1$. $\square$

We see that non-triviality of Chern classes gives an obstruction to triviality of the vector bundle.

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[3] Or other cohomology theories. For example, the Steifel–Whitney classes belong to $H^*(X;\mathbb{Z}/2)$, cohomology with $\mathbb{Z}/2$ coefficients.

[4] As previously mentioned, there is an abuse of notation here: for $c(f^*\nu)$, the f^* denotes the pullback operation on vector bundles. For $f^*c(\nu)$ the f^* is the induced map on cohomology.

Another simple consequence of the axioms is that the Chern classes are *stable*: $c(\nu) = c(\nu \oplus \xi)$ for any trivial bundle ξ. Indeed, by the Whitney sum formula and the above, $c(\nu) = c(\nu) \smile 1 = c(\nu) \smile c(\xi) = c(\nu \oplus \xi)$.

5. Cohomology of the classifying spaces and universal Chern classes

One elegant way of constructing the Chern classes is to just define them for the universal bundles. The rest of the Chern classes can then be defined as the pullbacks of the universal ones, from which one gets naturality for free. The other axioms come from nice algebraic structure on the universal Chern classes.

5.1. **Dimension 1.** In dimension 1 we have that $BU(1) \cong \mathbb{C}P^\infty$, infinite complex projective space. Its cohomology ring is $H^*(\mathbb{C}P^\infty) \cong \mathbb{Z}[c_1]$, where $|c_1| = 2$. It's not hard to compute the cohomology groups here (there's a cell decomposition with one cell in each even dimension). To get the ring structure one could use the Serre spectral sequence of the fibre bundle $S^1 \hookrightarrow S^\infty \to \mathbb{C}P^\infty$.

5.2. **Higher dimensions.** It turns out that $H^*(BU(r)) \cong \mathbb{Z}[c_1, c_2, \dots, c_r]$ where $|c_i| = 2i$. These generators c_i will later be called **universal Chern classes**.

For m, $n \in \mathbb{N}$ we have the diagonal embeddings $U(m) \times U(n) \hookrightarrow U(m+n)$, and these induce maps $w\colon BU(m) \times BU(n) \to BU(m+n)$ between the classifying spaces. In particular, we have the map induced from the embedding of the maximal torus:

$$d\colon BU(1)^r \to BU(r).$$

One may show that the induced map on cohomology

$$d^*\colon H^*(BU(r)) \to H^*(BU(1)^r) \cong \mathbb{Z}[t_1, t_2, \dots, t_r]$$

is injective (here each $|t_i| = 2$). Moreover, the symmetric group acts on $U(1)^r$ by permuting the coordinates, giving a symmetric action on $H^*(BU(1)^r)$ which permutes the generators t_i. So the image of d^* is contained in the symmetric polynomials of $\mathbb{Z}[t_1, \dots, t_r]$. One may show that in fact d^* is surjective onto them:

Theorem 5.1. *The map $d^*\colon H^*(BU(r)) \to H^*(BU(1)^r) \cong \mathbb{Z}[t_1, \dots, t_r]$ is an isomorphism onto its image, the sub-ring of symmetric polynomials in the t_i.*

So we have the following motto:

Elements of the cohomology ring $H^*(BU(r))$ can be identified with the symmetric polynomials of $H^*(BU(1)^r) \cong \mathbb{Z}[t_1, \dots, t_r]$.

Thinking ahead a little, if one has a sum of line bundles $\nu = \nu_1 \oplus \dots \oplus \nu_r$, then the axioms for the Chern classes demand that

$$c(\nu) = (1 + x_1)(1 + x_2) \cdots (1 + x_r),$$

where each x_i is a first Chern class $c_1(\nu_i)$. On the other hand, the kth Chern class of ν should be the degree k part of the polynomial above, which is the *elementary symmetric polynomial* in the x_i (the polynomial of the sum of each k-fold product of distinct x_i). Each generator t_i of $H^*(BU(1)^r)$ corresponds to a canonical generator of $H^*(BU(1))$, which will serve as the first universal Chern class. Therefore, we have a clear choice for the generators of $H^*(BU(r))$, resulting in a more specific motto to the one above:

The generator c_k of $H^*(BU(r)) \cong \mathbb{Z}[c_1, \dots, c_r]$ should to correspond to the kth elementary symmetric polynomial of $H^*(BU(1)^r) \cong \mathbb{Z}[t_1, \dots, t_r]$.

These elementary symmetric polynomials indeed generate the ring of all symmetric polynomials. The t_i are sometimes called **Chern roots**. They are often used in an unexplained formal sense, by identifying an expression of Chern roots t_i with an expression of the universal Chern classes c_i, even though they live in different cohomology rings (there is no element of $H^*(BU(3))$ corresponding to $t_1 - t_2^2 + t_1 t_3$, for example, this isn't a symmetric polynomial). But any polynomial in $H^*(BU(n)) = \mathbb{Z}[c_1, \dots, c_n]$ gives a polynomial in $H^*(BU(1)^n) = \mathbb{Z}[t_1, \dots, t_n]$ (by replacing each c_i with the kth symmetric polynomial in the t_i) and conversely any *symmetric* polynomial in the t_i has a unique expression in terms of the elementary symmetric polynomials, which we identify with the c_i.

The important relations between the universal Chern classes can be summarised as below. These properties follow reasonably quickly from Theorem 5.1 and our choices for the c_i in terms of the Chern roots. So most of the work here is in proving Theorem 5.1.

Definition 5.2. *For $r \geq 1$ we have* **universal Chern classes**

$$c_k^{(r)} = c_k \in H^{2k}(BU(r)),$$

and we define the **total universal Chern class** *to be*

$$c = c^{(r)} = 1 + c_1 + c_2 + \dots + c_r \in H^*(BU(r)).$$

These classes are characterised by the following axioms:

(1) $c_0 = 1$ *and* $c_k = 0$ *for* $k > r$;

(2) *for the canonical map* $i \colon BU(r) \to BU(r+1)$ *we have*

$$i^* c_k^{(r+1)} = c_k^{(r)};$$

(3) *for the canonical map* $w \colon BU(r_1) \times BU(r_2) \to B(r_1 + r_2)$ *we have*

$$w^* c^{(r_1+r_2)} = c^{(r_1)} \smile c^{(r_2)}$$

(technically, we mean the pullbacks of $c^{(r_1)}$ and $c^{(r_2)}$ to the product space);

(4) $c_1^{(1)}$ *is the canonical generator of* $H^2(BU(1)) \cong \mathbb{Z}$.

In the above, the map $BU(r) \to BU(r+1)$ comes from the canonical embedding $U(r) \hookrightarrow U(r+1)$ and corresponds to Whitney summing a trivial bundle. The space $BU(r_1) \times BU(r_2)$ should be thought of as classifying pairs of a rank r_1 and rank r_2 bundle, and the induced map to $BU(r_1 + r_2)$ is to be thought of as taking the Whitney sum and forgetting the decomposition as a pair.

6. Definition of the Chern Classes

We can now define the Chern classes of a general bundle using the universal Chern classes:

Definition 6.1. *Given $\nu \in \mathrm{Vect}_r^{\mathbb{C}}(X)$ and $f \colon X \to BU(r)$ classifying it (that is, $\nu = f^* \gamma_r$), we define the degree k* **Chern class** *as $c_k(\nu) = f^* c_k$, and the* **total Chern class** *to be*

$$c(\nu) = 1 + c_1(\nu) + c_2(\nu) + \dots + c_r(\nu) = f^* c.$$

Naturality of the Chern classes (Axiom 2 of Definition 4.1) follows directly from the above definition. Indeed, suppose that $g\colon X \to Y$ is a continuous map and ν is a vector bundle over Y, classified by the map $f\colon Y \to BU(r)$. Then $g^*\nu \cong g^*(f^*\gamma_r) = (f \circ g)^*\gamma_r$, so $g^*\nu$ is classified by the map $g \circ f$. Hence $c(g^*\nu) = (g \circ f)^*c = g^*(f^*c) = g^*c(\nu)$, as desired.

The other 3 axioms of the Chern class follow from the corresponding axioms of the universal Chern classes.

For example, the Whitney sum formula: Given a sum of a rank r_1 and rank r_2 bundles $\nu \oplus \xi$ classified by $f\colon X \to BU(n)$, one can factor f through $BU(r_1) \times BU(r_2)$, writing $f = w \circ (f_\nu \times f_\xi)$ (where f_ν and f_ξ classify ν and ξ, resp.). Then

$$f^*c = (w \circ (f_\nu \times f_\xi))^*c = (f_\nu \times f_\xi)^*(w^*c) = (f_\nu \times f_\xi)^*(c^{(r_1)} \smile c^{(r_2)}),$$

where $c = c^{(r_1+r_2)}$ and, technically, $c^{(r_i)} = \pi_i^* c^{(r_i)}$. The cup product above can be rewritten as the cup product of

$$(f_\nu \times f_\xi)^*(\pi_1^* c^{(r_1)}) = (\pi_1 \circ f_\nu \times f_\xi)^* c^{(r_1)} = f_\nu^* c^{(r_1)} = c(\nu)$$

with, similarly, $(f_\nu \times f_\xi)^*(\pi_2^* c^{(r_2)}) = c(\xi)$. So $c(\nu \oplus \xi) := f^*c = c(\nu) \smile c(\xi)$, as desired.

In addition to the nice Whitney sum formula for sums of bundles, you might hope that things work neatly with the tensor product too. They do, but only in degree one:

Theorem 6.2. *For two rank* 1 *bundles* ν *and* ξ *we have that*

$$c_1(\nu \otimes \xi) = c_1(\nu) + c_1(\xi).$$

The space $\mathbb{C}P^\infty \cong BU(1)$ can be given a CW decomposition and has homotopy groups concentrated in degree 2, where we have $\pi_2(BU(1)) \cong \mathbb{Z}$. As such, one calls $BU(1)$ an *Eilenberg–Mac Lane space* (usually denoted $K(\mathbb{Z}, 2)$). Just as one has classifying spaces to represent the functor $\mathrm{Vect}_n(-)$, cohomology is represented by the Eilenberg–Mac Lane spectrum, in particular we have natural identifications

$$\mathrm{Vect}_1^{\mathbb{C}}(X) \cong [X, BU(1)] \cong H^2(X)$$

for X a CW complex (or for weirder spaces if one replaces $H^*(-)$ with a more 'continuous' functor like Čech cohomology).

Theorem 6.3. *One has an isomorphism of groups, induced by taking the first Chern class:*

$$c_1\colon (\mathrm{Vect}_1^{\mathbb{C}}(X), \otimes) \xrightarrow{\cong} (H^2(X), +).$$

The bijection $[X, \mathbb{C}P^\infty] \to H^2(X)$ is given by $f \mapsto f^*(c_1)$, where c_1 is the universal Chern class, the generator of $H^*(\mathbb{C}P^\infty) \cong \mathbb{Z}[c_1]$, so $c_1(-)$ is a complete invariant for complex line bundles.

7. The Chern Character

Let X now be a compact space. We have the set $\mathrm{Vect}(X)$ of $\mathbb{C}$-vector bundles over X. Given two vector bundles ν and ξ, we can take their sum $\nu \oplus \xi$ as well as their tensor product $\nu \otimes \xi$. By taking formal differences (the 'group completion' or 'Grothendieck group') of $(\mathrm{Vect}(X), \oplus, \otimes)$, one gets a commutative ring, the **K-theory of** X, denoted $K^0(X)$.

It follows that for an element of $K^0(X)$ we can construct elements in $H^*(X)$, by taking some combination of Chern classes. What we'd really like is that doing so gives a ring homomorphism between K-theory and cohomology; we've already seen that taking Chern classes has some algebraic consistency with Whitney sums and tensor products (the Whitney sum formula and the above theorem). To construct an actual ring homomorphism we need to pass to $\mathbb{Q}$-coefficient cohomology, and then exponentiate:

Definition 7.1. *Let $t_1, \dots, t_r$ be the Chern roots for rank r bundles. We define the* **universal Chern class** *to be the element*

$$\mathrm{ch} := \exp t_1 + \exp t_2 + \cdots + \exp t_r \in \prod_{i=0}^{\infty} H^{2i}(BU(r); \mathbb{Q}),$$

where

$$\exp t_i = 1 + \sum_{i=0}^{\infty} t_i^k / k!.$$

Notice that the degree k part (times $k!$) of the above is $t_1^k + t_2^k + \cdots + t_r^k$. Since this is symmetric in the t_i, by convention it defines an expression in Chern classes. Indeed, in degree k,

$$t_1^k + t_2^k + \cdots + t_r^k = s_k(c_1, c_2, \dots, c_k),$$

where s_k is the *Newton polynomial* which expresses $t_1^k + t_2^k + \cdots + t_r^k$ as a polynomial in the elementary symmetric polynomials (corresponding to the c_i). For example,

$$s_1(x) = x, \quad s_2(x, y) = x^2 - 2y, \quad s_3(x, y, z) = x^3 - 3xy + 3z;$$

the elementary symmetric polynomials (for, say, $r = 3$) are

$$c_1 = t_1 + t_2 + t_3, \quad c_2 = t_1t_2 + t_1t_3 + t_2t_3, \quad c_3 = t_1t_2t_3, \quad c_k = 0, \ (k > 3)$$

and indeed

$$\begin{aligned} s_1(c_1) &= t_1 + t_2 + t_3, \\ s_2(c_1, c_2) = (t_1 + t_2 + t_3)^2 - 2(t_1t_2 + t_1t_3 + t_2t_3) &= t_1^2 + t_2^2 + t_3^2, \\ s_3(c_1, c_2, c_3) = (t_1 + t_2 + t_3)^3 - 3(t_1t_2 + t_1t_3 + t_2t_3) + 3(t_1t_2t_3) &= t_1^3 + t_2^3 + t_3^3. \end{aligned}$$

The Newton polynomials s_k don't actually depend on r, with the neat consequence that only the constant term below depends on the rank r:

$$\mathrm{ch} = r + \sum_{k=1}^{\infty} s_k(c_1, \dots, c_k)/k! = n + c_1 + (c_1^2 - c_2)/2 + (c_1^3 - 3c_1c_2 + 3c_3)/3! + \cdots .$$

Definition 7.2. *For a rank r vector bundle ν classified by $f \colon X \to BU(r)$ we define its* **Chern character** *to be*

$$\mathrm{ch}(\nu) := f^*(\mathrm{ch}) = r + \sum_{k=1}^{\infty} s_k(c_1(\nu), c_2(\nu), \dots, c_k(\nu))/k! \in H^{\mathrm{even}}(X; \mathbb{Q}) := \prod_{i=0}^{\infty} H^{2i}(X; \mathbb{Q}) .$$

One can also apply this formula to a formal difference of vector bundles, so to an element of K-theory. The beautiful thing is that this induces a well-defined ring homomorphism:

Theorem 7.3. *The Chern character defines a ring homomorphism*[5]

$$\mathrm{ch} \colon K^0(X) \to H^{\mathrm{even}}(X; \mathbb{Q}).$$

[5] Of course this is contained in $H^*(X; \mathbb{Q})$ (the direct sum) so long as $H^n(X; \mathbb{Q}) \cong 0$ for sufficiently large n. Our spaces are currently compact and all CW complexes have this property; if your space doesn't then you probably don't want to be looking at $H^*(X; \mathbb{Q})$.

That it is additive for direct sums of line bundles follows from the Whitney sum formula and that it is multiplicative follows from Theorem 6.3. Then ch is additive and multiplicative on all bundles by the 'Splitting Principle'[6]. Consequently it extends in a well-defined way to a ring homomorphism from $K^0(X)$. Amazingly, it is very close to an isomorphism, we have the *Chern isomorphism*:

Theorem 7.4. *By tensoring with $\mathbb{Q}$ the Chern character induces an isomorphism*

$$\mathrm{ch}\colon K^0(X)\otimes\mathbb{Q} \xrightarrow{\cong} H^{\mathrm{even}}(X;\mathbb{Q})$$

for finite CW complexes X.

The proof goes roughly as follows. Firstly, note that the Chern character is natural; this follows directly from the fact that the Chern classes are natural (commute with taking induced maps of conntiuous maps). The theorem obviously holds for the one-point space. It is then a general fact that a natural transformation between two cohomology theories which agree on the point agree on all finite CW complexes. This is because the identification of the cohomology with cellular cohomology only uses the Eilenberg–Steenrod axioms, so a calculation for one will reproduce the same calculation for the other.

Replacing H^* with a more continuous functor (sheaf or Čech cohomology) the above holds for more general kinds of compact spaces. By suspending we also get an isomorphism $K^1(X)\cong H^{\mathrm{odd}}(X;\mathbb{Q})$.

For finite CW complexes, the K-theory and cohomology groups are finitely generated free Abelian groups, so isomorphic to $\mathbb{Z}^k+T$ for some $k\in\mathbb{N}_0$ and finite torsion group T. The Chern isomorphism says that for $K^0\cong\mathbb{Z}^k\oplus T$ we may determine k as the sum of the ranks of the free parts of the even cohomology (and similar for K^1 and the odd cohomology groups). To say something more, on the torsion, one needs more complicated tools, such as the Atiyah–Hirzebruch spectral sequence. The torsion can indeed differ; for example $K^0(\mathbb{R}P^{2k})\cong\mathbb{Z}\oplus\mathbb{Z}/2^k$ but $H^{\mathrm{even}}(\mathbb{R}P^{2k})\cong\mathbb{Z}\oplus(\mathbb{Z}/2)^k$.

Appendix A. Complex bundles on spheres

We continue Example 2.9 on $\mathrm{Vect}_r^{\mathbb{C}}(S^n)\cong\pi_{n-1}(U(r))$. You'll need to know about fibre bundles and homotopy groups to make sense of most of this.

For $n=1$, since the unitary groups are connected (proven by the fact that matrices can be diagonalised), we have that $\pi_0(U(r))\cong 0$ so the only $\mathbb{C}$-vector bundles over S^1 are the trivial bundles.

For larger n, we consider the fibre bundle

$$U(r)\xrightarrow{\det} U(1),$$

where $U(1)\cong S^1$ and the above has fibres homeomorphic to $SU(r)$, the group of special unitary matrices (unitary matrices with determinant 1). This gives a long exact sequence of

[6] This says that, given a bundle ν over X, one may pull it back to a bundle $p^*\nu$ over some space Y so that $p^*\nu$ is a sum of line bundles and p induces an injection on cohomology. The moral is then that 'cohomological identities that hold for all sums of line bundles must also hold for general bundles'. Easy exercise: use this to show that the Chern classes are characterised by the axioms of Definition 4.1.

homotopy groups
(A.1)
$$\cdots \to 0 \to \pi_n(SU(r)) \xrightarrow{\cong} \pi_n(U(r)) \to 0 \to \cdots 0 \to \pi_1(SU(r)) \to \pi_1(U(r)) \to \pi_1(S^1) \cong \mathbb{Z} \to 1.$$
Now, a matrix of $SU(r)$ defines a point of $S^{2r-1} \subseteq \mathbb{C}^r$, by acting on a base point of the unit sphere. This is in fact a fibre bundle
$$SU(r) \to S^{2r-1}$$
with fibres homeomorphic to $SU(r-1)$ so we get a LES ending
$$\cdots \to \pi_1(SU(r-1)) \to \pi_1(SU(r)) \to \pi_1(S^{2r-1}) \to 0.$$
For $r = 2$, $SU(1)$ is the trivial group and $SU(2) \cong S^3$, so $\pi_1(SU(2)) \cong 0$ as the spheres are simply connected: $\pi_n(S^k) \cong 0$ for all $n < k$. From the LES above, we see by induction that each $SU(r)$ is also simply connected. From the LES of Equation A.1 we see that we have bijections:
$$\mathrm{Vect}_r^{\mathbb{C}}(S^2) \cong \pi_1(U(r)) \cong \mathbb{Z},$$
For $r = 1$, $\mathrm{Vect}_1^{\mathbb{C}}(S^2)$ is a group with addition given by taking the tensor product of line bundles (c.f., Theorem 6.3) and the above is actually an isomorphism of groups. It doesn't take much more work to show that the maps $BU(r) \to BU(r+1)$ given by the diagonal embedding induce an isomorphism on π_1. This corresponds to taking the Whitney sum with a trivial bundle, so in fact: *every $\mathbb{C}$-bundle over S^2 is a sum of copies of a single line bundle γ, or its dual bundle, and the rank 1 trivial bundle.* One may also prove that $\gamma \oplus \gamma = (\gamma \otimes \gamma) \oplus 1$, where 1 is the rank 1 trivial bundle, which completely describes $(\mathrm{Vect}^{\mathbb{C}}(S^2), \oplus, \otimes)$.

For higher dimensional spheres things are *a lot* more complicated; one needs to calculate the higher homotopy groups of $U(r)$ (equivalently $SU(r)$). One can find tables of these for low enough values; for example: over the S^5 there is 1 rank one $\mathbb{C}$-bundle, 12 rank two bundles, 6 rank three bundles and just a trivial rank four bundle. Classifying $\mathbb{R}$-bundles is even more difficult.