INTRODUCTION TO (OPERATOR ALGEBRA) K-THEORY, JAMIE WALTON,

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1. Origins

K-theory for topological spaces was developed by Atiyah and Hirzebruch for topological spaces in 1959, with roots in earlier work by Grothendieck in algebraic geometry. K-theory for C^*-algebras began to be developed in the 1970s.

Although our interest is in C^*-algebras, let's very briefly see how the construction goes for topological spaces (there are important relations between the two approaches). Let X be a compact, Hausdorff space; a minor modification is needed if the space is only LCH = locally compact, Hausdorff. One may associate to X the set $\mathrm{Vect}_{\mathbb{C}}(X)$ of isomorphism classes of complex vector bundles over X. Two vector bundles can be Whitney summed $E_1 \oplus E_2$, making $(\mathrm{Vect}(X), \oplus)$ a commutative monoid (the unit being the trivial 0-dimensional bundle over X). Taking the Grothendieck completion of formal differences (which shall be covered later) produces an Abelian group $K^0(X)$. In fact, there are higher K-groups: $K^1(X)$, $K^2(X)$, $K^3(X)$, $\ldots$, making K^* a cohomology theory. A characteristic feature of K-theory is *Bott periodicity*: $K^i(X) \cong K^{i+2}(X)$ for all i, so in a sense we only have two K-groups to consider: $K^0(X)$ and $K^1(X)$.

As we shall see the construction of the K-groups for a C^*-algebra A follows a similar process (in a way which can be made precise, Swan's Theorem). Vector bundles over X are replaced with projections in the matrix algebras $M_m(A)$. When $A = C_0(X)$, the K-theory of the C^*-algebra is naturally isomorphic to the topological K-theory of X.

2. Key Properties: K-theory as a homology theory

One may often use K-theory as a 'black box', utilising its powerful key properties rather than computing directly from its definition. So we first list some useful such properties, similar to the Eilenberg–Steenrod axioms from topology.[1]

2.1. **Functorality.**

For each $n \in \mathbb{Z}$, taking K-theory in degree n defines a covariant functor

$$K_n \colon \mathscr{C}^* \to \mathscr{A}$$

from the category $\mathscr{C}^*$ of C^*-algebras and the category $\mathscr{A}$ of Abelian groups. Being a **covariant functor** here means the following:

(1) Given a C^*-algebra A, applying K_n gives an Abelian group $K_n(A)$.

[1]There is the dual theory to K-theory for C^*-algebras, called K-homology, so I may be using non-standard terminology in this context when listing the properties of K-theory 'as a homology theory'. I'm just saying 'homology' here because K-theory is a covariant (see below) functor for C^*-algebras.

(2) Given a $*$-homomorphism (= *homomorphism*, in these notes) $f\colon A \to B$ of C^*-algebras we have an **induced map** $f_*\colon K_n(A) \to K_n(B)$, a homomorphism between[2] the K-groups of A and B.

(3) These assignments respect the categorical structures of $\mathscr{C}^*$ and $\mathscr{A}$. That is:
- Identities are respected: if $\mathrm{id}_A\colon A \to A$ is an identity morphism then $(\mathrm{id}_A)_* = \mathrm{id}_{K_n(A)}$, the identity morphism on $K_n(A)$.
- Composition is respected: if $f\colon A \to B$ and $g\colon B \to C$ then $(g \circ f)_* = g_* \circ f_*$.

2.2. **Homotopy Invariance.** We call two homomorphisms $f,\ g\colon A \to B$ of C^* algebras **homotopic** and write $f \sim g$ if there is a continuous path $f_t\colon [0,1] \times A \to B$ of homomorphisms with $f_0 = f$ and $f_1 = g$ (continuity here means that for any $a \in A$, the map $t \mapsto f_t(a)$ is continuous from $[0,1]$ to B).

K-theory is homotopy invariant: If $f,\ g\colon A \to B$ are homotopic then $f_* = g_*$.

This implies, for example, that homotopy equivalent C^*-algebras have isomorphic K-theory (we call A and B **homotopy equivalent** if there exist homomorphisms $f\colon A \to B$ and $g\colon B \to A$ with $g \circ f \sim \mathrm{id}_A$ and $f \circ g \sim \mathrm{id}_B$).

Exact sequences. Before stating the next property, recall that a pair of maps

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is called **exact at** B if $\mathrm{im}(f) = \ker(g)$ (we assume that A, B, C, f and g are objects/morphisms in a suitable algebraic category where images and kernels make sense, for us C^*-algebras and Abelian groups). A larger diagram is called **exact** if it is exact at each position where this makes sense. For example, the following sequence

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0,$$

being exact means that f is injective, g is surjective and $\mathrm{im}(f) = \ker(g)$. An exact sequence of this particular size is called a **short exact sequence**. An exact sequence

$$\cdots \to A_{-1} \to A_0 \to A_1 \to A_2 \to A_3 \to \cdots$$

is called a **long exact sequence**.

Up to isomorphism a short exact sequence is of the form

$$0 \to I \hookrightarrow B \xrightarrow{q} B/I \to 0$$

where $I \leqslant B$, the first map is the inclusion and the second is the quotient. For C^*-algebras, I here is a (closed, two-sided) ideal of B. In a short exact sequence of groups I is a normal subgroup of B.

[2]Being 'covariant' means that the induced map goes $f_*\colon K_n(A) \to K_n(B)$. The alternative would to be a *contravariant functor*, which flips the direction of the induced map. For example, topological K-theory is contravariant: if we have a continuous map $f\colon X \to Y$ between Hausdorff, compact spaces, then we get an induced homomorphism of Abelian groups $f^*\colon K^n(Y) \to K^n(X)$. This makes sense, the category of commutative, unital C^*-algebras is isomorphic to the *opposite* category of compact, Hausdorff spaces. That K-theory is covariant for C^*-algebras and contravariant for spaces is indicated by the lower (K_n) versus upper (K^n) indexing.

2.3. **Long exact sequence of an extension.** Given a short exact sequence

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

of C^*-algebras there is an induced long exact sequence

$$\cdots \xrightarrow{g_*} K_{-1}(C) \xrightarrow{\partial} K_0(A) \xrightarrow{f_*} K_0(B) \xrightarrow{g_*} K_0(C) \xrightarrow{\partial} K_1(A) \xrightarrow{f_*} K_1(B) \xrightarrow{g_*} K_1(C) \xrightarrow{\partial} \cdots .$$

There is some notational laziness here, since we've denoted the maps $\partial\colon K_n(C) \to K_{n+1}(A)$ identically, without regard for n (of course, we've already done this for the induced maps f_*, g_* too).

By Bott periodicity (a 'special' property of K-theory, so we will list it later) the above infinite long exact sequence can be conveniently repackaged as a six-term exact sequence:

$$\begin{tikzcd} K_0(A) \arrow[r, "f_*"] & K_0(B) \arrow[r, "g_*"] & K_0(C) \arrow[d, "\partial_0"] \\ K_1(C) \arrow[u, "\partial_1"] & K_1(B) \arrow[l, "g_*"] & K_1(A) \arrow[l, "f_*"] \end{tikzcd}$$

This famous six-term sequence is the source of much of the power of K-theory. A short exact sequence of C^*-algebras is, up to isomorphism, the inclusion of an ideal I into B, followed by the quotient map. So the above means that you can say something about the K-theory of one of I, B or B/I whenever you have some information on the K-theory of the others, together with information on how these are connected by the induced/connecting maps (having trivial groups around often saves the day).

The map ∂_0 above is called the **exponential map**, and ∂_1 is called the **index map**.

2.4. **Additivity.** Given a set $\{A_i \mid i \in \mathcal{I}\}$ of C^*-algebras, there is a natural isomorphism

$$K_n\left(\bigoplus_{i\in\mathcal{I}} A_i\right) \cong \bigoplus_{i\in\mathcal{I}} K_n\left(A_i\right).$$

That is: "the K-theory of a direct sum of C^*-algebras is the direct sum of their K-theories".

Later, we shall see that K-theory has a related *continuity property.*

2.5. **Excision.** I won't cover this one because I don't have the time to cover the relative K-groups (which take a pair $(A, A/I)$ for a C^* algebra A with ideal $I \leqslant A$). It's usual to introduce the relative K-groups in defining the K-theory of non-unital C^*-algebras. Excision theorems allow one to replace relative K-groups with the usual, non-relative ones. However, it is sometimes more natural to express certain objects as elements of relative K-groups; again, we'll pass by this in this introduction.

3. Other important properties

The above properties are what make K-theory a 'homology theory' for C^*-algebras. But it has some other important properties too, some of which we've already mentioned:

3.1. **Bott Periodicity.** For all $n \in \mathbb{Z}$ we have a natural isomorphism $K_n(A) \cong K_{n+2}(A)$. Being 'natural' means that the isomorphisms respect the extra structure of induced maps and connecting maps of the long exact sequence. In particular, it allows us to derive the six-term exact sequence from the long exact sequence.

3.2. **Suspension.** This really belongs in the last section, as a property of a homology theory, but I wanted to mention Bott periodicity first. We have natural isomorphisms

$$K_{n+1}(A) \cong K_n(SA)$$

where SA is the suspension of A. Given a C^*-algebra A, one may define its **suspension** as the C^*-algebra

$$SA := \{\phi \in C([0,1], A) \mid \phi(0) = \phi(1) = 0\}. \tag{3.1}$$

Alternatively, you can of course think instead of functions $\phi \in C(S^1, A)$ sending some fixed basepoint of the circle[3] to 0. Or you could think of continuous functions $\phi\colon \mathbb{R} \to A$ vanishing at infinity.

Suspension is a covariant functor: given a homomorphism $f\colon A \to B$ of C^*-algebras, we have a homomorphism $Sf\colon SA \to SB$ between their suspensions. It's obvious how to do this: given $\phi \in (C[0,1], A)$ just compose with f, so $Sf(\phi) := f \circ \phi \in SB$.

So one could define the higher K-groups inductively, starting with K_0 and inductively defining $K_{n+1}(A) := K_n(SA)$. In fact, this is what one does. That shouldn't disappoint you, since by Bott periodicity we only have to worry about two different K-groups K_0 and K_1, and both of these have direct definitions. In particular, there is content here for K_1: one may equivalently define it as $K_1(A) := K_0(SA)$ or in a direct fashion in terms of unitaries as will be done at the end of these notes.

3.3. **Theorems for cross-products.** We have the following two important theorems on the K-theory of cross-product C^*-algebras (if you know about these):

Theorem 3.1 (Pimsner–Voiculescu)**.** *There is a six-term exact sequence*

$$\begin{tikzcd}
K_0(A) \arrow[r, "(\mathrm{id}_A)_* - (\alpha(-1))_*"] & K_0(A) \arrow[r, "i_*"] & K_0(A \rtimes_\alpha \mathbb{Z}) \arrow[d] \\
K_1(A \rtimes_\alpha \mathbb{Z}) \arrow[u] & K_1(A) \arrow[l, "i_*"] & K_1(A) \arrow[l, "(\mathrm{id}_A)_* - (\alpha(-1))_*"]
\end{tikzcd}$$

The above says that you can say something about the K-theory $A \rtimes_\alpha \mathbb{Z}$ if you can say something about the K-theory of A along with the map on K-theory induced by the generator of the $\mathbb{Z}$-action.

For a cross-product with $\mathbb{R}^d$ there is the following *Thom–Connes isomorphism*:

Theorem 3.2 (Connes)**.** *There is an isomorphism*

$$K_n(A \rtimes_\alpha \mathbb{R}^d) \cong K_{n+d}(A).$$

3.4. **Continuity.** Consider a sequence

$$A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \xrightarrow{f_3} A_4 \xrightarrow{f_4} A_5 \xrightarrow{f_5} \cdots \tag{3.2}$$

of C^*-algebras and homomorphisms between them. Since K-theory is functorial, we can plug this whole diagram into K_n and get a diagram of Abelian groups and homomorphisms:

$$K_n(A_1) \xrightarrow{(f_1)_*} K_n(A_2) \xrightarrow{(f_2)_*} K_n(A_3) \xrightarrow{(f_3)_*} K_n(A_4) \xrightarrow{(f_4)_*} K_n(A_5) \xrightarrow{(f_5)_*} \cdots . \tag{3.3}$$

[3]This definition looks more like the the analogue from topology of the loop space functor rather than suspension, which is kind of the opposite (the two are adjoint functors). But again, topologists need to think upside down: the opposite category.

Continuity says that we have a natural isomorphism:

$$K_n(\varinjlim_{i\in\mathbb{N}} A_i) \cong \varinjlim_{i\in\mathbb{N}} K_n(A_i),$$

where the left-hand side is the K-theory of the inductive limit of C^*-algebras of Equation 3.2, and the right-hand side is the inductive limit of groups from Equation 3.3. That is: "the K-theory of an inductive limit of C^*-algebras is the inductive limit of K-theories". Or: "K-theory commutes with limits" (hence the name *continuity*). There is continuity over more general kinds of limit diagrams of C^*-algebras too.

I won't give full details on what an inductive limit is. It may be defined in a standard categorical way via a universal property, and also in a more direct fashion. A useful case to have in mind is when each f_i is an inclusion of C^*-algebras, in which case:

$$\varinjlim_{i\in\mathbb{N}} A_i \cong \overline{\bigcup_{i=1}^{\infty} A_i}.$$

Of course the union is meant as a nested, rather than disjoint union.

For a diagram of groups the inductive limit's elements are represented by elements of the disjoint union of groups, where we identify elements which are eventually mapped to the same element. The group operation is given by sending two representative group elements forward to a common group and multiplying them there.

3.5. **Stability.** It won't be too hard to see directly from the definitions to follow that for integers $1 \leq m \leq m'$ the obvious inclusion

$$M_m(A) \hookrightarrow M_{m'}(A)$$

induces an isomorphism

$$K_n(M_m(A)) \cong K_n(M_{m'}(A)).$$

In particular we have an isomorphism

$$K_n(A) \cong K_n(M_m(A)).$$

It follows from continuity that for the inductive limit C^*-algebra

$$\mathcal{K}A := \varinjlim(M_1(A) \hookrightarrow M_2(A) \hookrightarrow M_3(A) \hookrightarrow \cdots) \cong \overline{\bigcup_{m=1}^{\infty} M_m(A)} = \overline{M_\infty(A)}$$

the natural map $A \to \mathcal{K}A$ induces an isomorphism

$$K_n(A) \cong K_n(\mathcal{K}A).$$

That is, K-theory is *stable*.

The C^*-algebra $\mathcal{K}A$ is called the **stabilisation of** A, and it may be shown that

$$\mathcal{K}A \cong \mathcal{K} \otimes A,$$

where $\mathcal{K}$ on the right-hand side is the C^*-algebra of compact operators on a separable, infinite dimensional Hilbert space.

4. Definition of K_0

Enough of the properties, what are the K-groups?

To make things simpler, we will assume now that our C^*-algebra A is unital. A small modification is needed in the non-unital case.

Recall that an element $p \in A$ is a **projection** (or self-adjoint idempotent) if

$$p^* = p = p^2.$$

Let $\mathrm{Proj}(A)$ be the set of projections. We say that two projections are **orthogonal**, and write $p \perp q$, if $pq = 0$ (equivalently, $qp = 0$).

We call p and q **Murray von Neumann equivalent**, and write $p \sim_{mvn} q$, if there exists some $v \in A$ with $v^*v = p$ and $vv^* = q$. Sometimes one calls v a **partial isometry** from p to q.

Naïvely we would like to define a group operation on $\mathrm{Proj}(A)/\sim_{mvn}$ by:

$$[p] + [q] = [p + q].$$

This is too simplistic; $p + q$ need not even be a projection! It would be if $p \perp q$ but in general it need not be the case that we can find orthogonal representatives of p and q.

We give ourselves more space by passing to $M_\infty(A)$. It was already mentioned earlier but let's be more precise about its definition. Consider the inclusions $M_m(A) \hookrightarrow M_{m+1}(A)$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1m} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2m} \\ \cdots & \cdots & \cdots & \ddots & \cdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mm} \end{pmatrix} \mapsto \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1m} & 0 \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2m} & 0 \\ \cdots & \cdots & \cdots & \ddots & \cdots & 0 \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mm} & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

We let $M_\infty(A)$ be the corresponding 'algebraic direct limit' (the nested union, but we don't take a completion at the end). Equivalently, one may identify the elements of $M_\infty(A)$ with infinite matrices of elements in A for which all but finitely many entries are 0. An element of $M_m(A)$ is identified with such a matrix in the obvious way, placing it in the top-left corner. For $a, b \in M_\infty(A)$ the elements $a+b$, ab and a^* are still defined (applying these operations in some $M_m(A)$, or thinking of the elements as infinite matrices). It is not quite a C^*-algebra, since we haven't defined the norm and then taken the completion (which would give $\mathcal{K}A$).

Given $p \in M_m(A)$, $q \in M_n(A)$, we define $\mathrm{diag}(p, q) := \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} \in M_{m+n}(A)$, with each '0' the appropriately sized block of 0s.

We still have a notion of projection in $M_\infty(A)$ and of two elements being MVN equivalent. We define:

$$V(A) := \mathrm{Proj}(M_\infty(A))/\sim_{mvn}.$$

We are now in the fortunate situation that any two $p, q \in \mathrm{Proj}(M_\infty(A))$ are MVN equivalent to projections which are orthogonal. Indeed, if $p \in M_m(A)$ and $q \in M_m(A)$, then think of them as both elements of $M_{2m}(A)$. Move q down the diagonal: $q \sim \mathrm{diag}(0_m, q)$, where 0 denotes the 0-matrix in $M_m(A)$. Indeed, just let $v = \begin{pmatrix} 0 & q \\ 0 & 0 \end{pmatrix}$, then $v^*v = \mathrm{diag}(q, 0_m)$ and $v^*v = \mathrm{diag}(0_m, q)$. Obviously p and $\mathrm{diag}(0_m, q)$ are orthogonal, considered as elements of $M_\infty(A)$.

So we may define an addition on $\mathrm{Proj}(V(A))/\sim_{mvn}$ by letting

$$[p] + [q] \coloneqq [p + q]$$

where we choose representatives p and q for which $p \perp q$.

Proposition 4.1. *The above makes $(V(A), +)$ an Abelian semigroup with identity (a commutative monoid). That is, $+$ is a well-defined, commutative, associative binary operation on $V(A)$. The identity element is represented by the 0-matrix.*

To get a *group*, rather than just a monoid (which lacks inverses), we take the *Grothendieck completion*, aka the associated **Grothendieck group**: Given a commutative monoid $(M, +)$, the elements of the Grothendieck group $G(M, +)$ are represented by elements $(g_1, g_2) \in M \times M$, which we think of as formal differences "$g_1 - g_2$". We identify (g_1, g_2) and (h_1, h_2) if

$$g_1 + h_2 + k = g_2 + h_1 + k$$

for some $k \in M$. The $+k$ is necessary in the case where M doesn't have cancellation i.e., when it does not hold that

$$g + k = h + k \Rightarrow g = h.$$

Addition is defined in $G(M, +)$ by:

$$[g_1, g_2] + [h_1, h_2] \coloneqq [g_1 + h_1, g_2 + h_2],$$

think: "$(g_1 - g_2) + (h_1 - h_2) = (g_1 + h_1) - (g_2 + h_2)$". The Grothendieck group may be defined via a universal property; you should think of it as the minimal way of constructing a group out of a monoid.

Example 4.2. $G(\mathbb{N}_0) = \mathbb{Z}$, with $\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$ and $\mathbb{Z}$ equipped with standard addition.

Definition 4.3. *For a unital C^*-algebra A we define $K_0(A) \coloneqq G(V(A), +)$.*

In summary: $K_0(A)$ is the Grothendieck completion of the semigroup $\mathrm{Proj}(M_\infty(A))/\sim_{mvn}$, where addition is defined by addition of orthogonal representatives.

4.1. Examples.

Example 4.4. Let $A = \mathbb{C}$. We claim that taking the rank defines an isomorphism

$$\mathrm{rk}\colon (V(A), +) \xrightarrow{\cong} (\mathbb{N}_0, +).$$

Indeed, taking the rank is well defined on MVN classes: $\mathrm{rk}(v^*v) = \mathrm{rk}(vv^*)$ (simple exercise in linear algebra). The rank of a sum of orthogonal projections is the sum of the ranks (i.e., the dimension of the image). Obviously rk is onto $\mathbb{N}_0$.

It is injective because two projections of the same rank are related by a partial isometry. In more detail, let p, $q \in M_m(A)$ be of rank r. Since p is a projection we may diagonalise it as $p = udu^*$, where $d = \mathrm{diag}(\mathrm{id}_k, 0_{m-k})$ and u is a unitary (that is, $u \in M_m(A)$ with $u^*u = 1 = uu^*$). Similarly $q = vdv^*$. So

$$p = u(d)u^* = (ud^*)(du^*) \sim_{mvn} (du^*)(ud^*) = dd^* = d.$$

Analogously $q \sim_{mvn} d$ and hence $p \sim_{mvn} q$.

Since $(V(A), +) \cong (\mathbb{N}_0, +)$ we have that $K_0(\mathbb{C}) \cong \mathbb{Z}$.

Example 4.5. Let $\mathcal{K}$ be the compact operators in $B(\mathcal{H})$, with $\mathcal{H}$ an infinite dimensional, seperable Hilbert space. By stability $\mathcal{K} = \mathcal{K}\mathbb{C}$ has the same K-theory as $\mathbb{C}$.

Example 4.6. One may show that for $\mathcal{H}$ as above, $V(B(\mathcal{H})) \cong \mathbb{N}_0 \cup \{\infty\}$, with standard addition extended from $\mathbb{N}_0$ in the obvious way (that is, $\infty + n = n + \infty = \infty$ for any n). It's essentially the same idea to Example 4.4, but in $B(\mathcal{H})$ projections can have infinite rank. The Grothendieck completion of this is trivial: $x + \infty = y + \infty$ for any x, y, so cancellation fails so horribly that all elements are identified. Hence $K_0(B(\mathcal{H})) = 0$.

Example 4.7. Let CA be the **cone** of A:

$$CA = \{\phi \in C([0, 1], A) \mid \phi(0) = 0\}.$$

Compare with Equation 3.1.

The cone CA is homotopy equivalent to the trivial C^*-algebra 0 of one element, via the zero homomorphisms

$$f \colon CA \to 0; \; g \colon 0 \to CA.$$

Indeed, $f \circ g$ is already the identity map on 0, and $g \circ f = (\phi \mapsto 0)$ is homotopic to the identity map via $\psi_t(\phi)(s) := \phi(t \cdot s)$. So $K_n(CA) \cong K_n(0)$.

Clearly $K_0(0) \cong 0$. Once K_1 has been defined it will be clear that $K_1(0) \cong 0$ too.

Example 4.8. The cone and suspension fit into a short exact sequence

$$0 \to SA \to CA \to A \to 0.$$

The first map is the inclusion, the second sends a function $\phi \in CA$ to $\phi(1) \in A$.

Since the cone has trivial K-theory, the six-term exact sequence gives:

$$\begin{tikzcd}
K_0(SA) \arrow[r, "f_*"] & 0 \arrow[r, "g_*"] & K_0(A) \arrow[d, "\partial_0"] \\
K_1(A) \arrow[u, "\partial_1"] & 0 \arrow[l, "g_*"] & K_1(SA) \arrow[l, "f_*"]
\end{tikzcd}$$

So the exponential map induces an isomorphism $K_0(A) \cong K_1(SA)$ and the index map induces an isomorphism $K_1(A) \cong K_0(SA)$.

4.2. **Alternative constructions of K_0.** There are other equivalence relations on the projections $\mathrm{Proj}(M_\infty(A))$:

(1) **Homotopy equivalence:** $p \sim_h q$ if there is a path of projections in connecting p and q.

(2) **Unitary equivalence :** $p \sim_u q$ if there exists a unitary u in some $M_m(A)$ with $p = uqu^*$.

Whilst these equivalence relations don't quite agree on a fixed C^*-algebra, they all agree on $M_\infty(A)$ and so lead to the same semigroup $V(A)$ and therefore may be used instead[4] to define K_0. One can also replace $M_\infty(A)$ with its completion, the C^*-algebra $\mathcal{K}A = \mathcal{K} \otimes A$.

[4] I believe that one still gets the same result by using algebraic equivalence or similarity, defined below. But it is probably unnatural to consider these equivalence relations on the projections.

Instead of using projections one may use idempotents on $M_\infty(A)$, identifying by one of:

(1) **Algebraic equivalence:** $p \sim q$ if there exist x, y with $xy = p$, $yx = q$.

(2) **Homotopy equivalence:** $p \sim_h q$ if there is a path of idempotents connecting p and q.

(3) **Similarity:** $p \sim_s q$ if there exists an invertible s in some $M_m(A)$ with $p = sqs^{-1}$.

Taking the idempotents modulo any of these relations results in the same semi-group to before, so one may define $K_0(A)$ this way instead. Note that one doesn't need A to be a $*$-algebra for this to make sense, so one could apply this to a general Banach algebra (in fact, it works for mere 'local Banach algebras').

5. Definition of K_1

Let $U_m(A)$ denote the group of unitaries of $M_m(A)$. We may define $U_\infty(A)$ similarly to $M_\infty(A)$, where we embed $U_m(A) \hookrightarrow U_{m+1}(A)$ by $q \mapsto \mathrm{diag}(q, 1)$. Elements of $U_\infty(A)$ may be thought of as infinite unitary matrices whose diagonal elements are eventually 1 and all but finitely many entries off the diagonal are 0.

We let $U_m(A)_0$ be the path-component of the identity in $U_m(A)$ (allowing $m = \infty$).

Definition 5.1. $K_1(A) \cong U_\infty(A)/U_\infty(A)_0$.

The inclusion $U_m(A) \hookrightarrow U_{m+1}(A)$ maps $U_m(A)_0$ into $U_{m+1}(A)_0$ and one may alternatively write $K_1(A) \cong \varinjlim(U_m(A)/U_m(A)_0)$. Moreover, one may use invertibles in place of unitaries and get the same group, so again the construction can be made to work for general Banach algebras. It is not hard to show (but there is something to check) that $K_1(A)$ is an Abelian group.

5.1. Examples.

Example 5.2. The group $U_m(\mathbb{C})$ of unitary matrices is path-connected: any $u \in U_m(\mathbb{C})$ can be diagonalised to $v \cdot \mathrm{diag}(z_1, z_2, \ldots, z_m) \cdot v^*$ where $v \in U_m(\mathbb{C})$ and each $z_j \in \mathbb{C}$ with $|z_j| = 1$. So we may write $z_j = e^{i\theta_j}$. A path from u to the identity is then given by $u_t = v \cdot \mathrm{diag}(e^{it\theta_1}, e^{it\theta_2}, \ldots, e^{it\theta_m}) \cdot v^*$.

It follows that $U_\infty(\mathbb{C})/U_\infty(\mathbb{C})_0$ is the trivial group, so $K_1(\mathbb{C}) \cong 0$.

By stability we also have that $K_1(M_m(\mathbb{C})) \cong 0$ for any $m \in \mathbb{N}$, and $K_1(\mathcal{K}) \cong 0$. Since every finite-dimensional C^*-algebra F is isomorphic to a direct sum of matrix algebras $M_m(\mathbb{C})$, it follows from additivity that $K_1(F) \cong 0$. By continuity, every AF algebra (inductive limit of finite dimensional algebras) has trivial K_1.

Example 5.3. Let $\mathcal{H}$ be an infinite-dimensional, separable Hilbert space. It turns out that $U_m(B(\mathcal{H}))$ is path-connected, so $K_1(B(\mathcal{H})) \cong 0$.

We have an exact sequence

$$0 \to \mathcal{K} \to B(\mathcal{H}) \to B(\mathcal{H})/\mathcal{K} \to 0.$$

The quotient group on the right is called the **Calkin algebra**. Taking the six-term exact sequence and our previous calculations:

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, "f_*"] & 0 \arrow[r, "g_*"] & K_0(B(\mathcal{H})/\mathcal{K}) \arrow[d, "\partial_0"] \\
K_1(B(\mathcal{H})/\mathcal{K}) \arrow[u, "\partial_1"] & 0 \arrow[l, "g_*"] & 0 \arrow[l, "f_*"]
\end{tikzcd}$$

So $K_0(B(\mathcal{H})/\mathcal{K}) \cong 0$ and the index map gives an isomorphism

$$\partial_1 \colon K_1(B(\mathcal{H})/\mathcal{K}) \xrightarrow{\cong} \mathbb{Z}.$$

The study of this kind of thing is heading towards index theory, Fredholm operators,...

References

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