# Topological Methods in Aperiodicity, Lecture Summary and Exercises

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# Contents

**Abstract**

Notes and exercises to supplement the mini-course *Topological Methods in Aperiodicity*, Universität Wien, 7–9 June 2017. For time and space's sake, there will be many omissions here: several covered in the lectures, a few unintended, some strategically intended, and many regretfully intended. For many of these gaps, there is plenty of good literature to consult (please ask, or see the bibliography). Much of this material can be found in Sadun's book [27], which is good starting place for any newcomer to the field.

# Session 1

# Aperiodic order

## 1.1 Some motivating examples of aperiodic order

### 1.1.1 Beatty sequences and Sturmian sequences

Take an irrational number $\theta > 0$ and consider the sequence of integers

$$(B_n)_{n\in\mathbb{N}} = \lfloor\theta\rfloor, \lfloor 2\cdot\theta\rfloor, \lfloor 3\cdot\theta\rfloor, \lfloor 4\cdot\theta\rfloor, \lfloor 5\cdot\theta\rfloor, \ldots,$$

where $\lfloor x\rfloor = \max\{n \in \mathbb{Z} \mid n \leq x\}$ denotes the floor of $x$. Such a sequence is called a *Beatty sequence*. The values of $(B_n)_{n\in\mathbb{N}}$ jump by integer values of either $\lfloor\theta\rfloor + 1$ or $\lfloor\theta\rfloor$; denote these jumps by $a$ and $b$, respectively. This defines a new sequence $(S_n)_{n\in\mathbb{N}}$ on the alphabet $\{a, b\}$. For example, for $\theta = (1 + \sqrt{5})/2$ the golden ratio, we get the sequence:

$$a, b, a, a, b, a, b, a, a, b, a, a, b, \ldots.$$

Sequences formed in this way have some very interesting properties. They are highly 'repetitive': any finite sub-word (i.e., a finite sequence of letters $S_l, S_{l+1}, S_{l+2}, \ldots, S_{r-1}, S_r$) will reappear with some frequency across the entire word; in particular it will be found in any sub-word of a sufficiently large length (in fact, for the *Fibonacci sequence* coming from the golden ratio, there exists a $C > 0$ so that any sub-word of length $n$ occurs inside any other of length $C \cdot n$). On the other hand, it is not '(eventually) periodic': there does not exist some $n \in \mathbb{N}$ for which $S_k = S_{k+n}$ for all sufficiently large $k \in \mathbb{N}$. Sequences coming from this construction turn out to have the smallest number of sub-words possible for non-periodic sequences: there are precisely $n+1$ sub-words of length $n$ for all $n \in \mathbb{N}$. Such words are called *Sturmian* (in fact, essentially all Sturmian sequences come from a construction like this one [11]).

A periodic sequence would automatically be repetitive, and there is little to discover about its structure, being such a simple sequence (a concatenation $vvv\ldots$ for some finite word $v$). We are interested in patterns like the Fibonacci sequence above, which are clearly highly structured—say, because they are highly repetitive—and yet are not periodic.

### 1.1.2 Cut-and-project sets

There is an alternative construction of the Sturmian words—via the *cut-and-project method*—which is highly generalisable (see also the approach via *cutting sequences* [29]). In fact, we shall look only at a special case of the construction, but we shall indicate how one generalises further. We start with the following data:

- a Euclidean space $\mathbb{R}^k$ called the *total space*;
- a subspace $E$ of $\mathbb{R}^k$ of dimension $d$ with $0 < d < k$, called the *physical space*;
- a subset $\mathcal{W} \subset \mathbb{R}^k$ called the *window*.

The window is typically a simple subset, like a hypercube. This defines the *strip* $\mathcal{S} \coloneqq E + \mathcal{W}$, which we think of as a thickened version of $E$. Given $s \in \mathbb{R}^k$, the *lifted cut-and-project set* $\tilde{Y}_s$ associated to all of this data is given by:

$$\tilde{Y}_s \coloneqq (\mathbb{Z}^k + s) \cap \mathcal{S},$$

i.e., it is given by elements of the shifted lattice $\mathbb{Z}^k + s$ which fall into the strip. We define the *cut-and-project set* $Y_s$ by projecting $\tilde{Y}_s$ onto $E$, say via orthogonal projection (one could generalise mildly here and allow different projections. The choice of projection typically does not play a large rôle).

Most subspaces $E$ can be given as the graph of a family of linear forms $L_i \colon \mathbb{R}^d \to \mathbb{R}$, for $i = 1, \ldots, k-d$. We set $L(x) \coloneqq (L_1(x), \ldots, L_{k-d}(x)) \in \mathbb{R}^{k-d}$ for $x \in \mathbb{R}^d$, and then

$$E \coloneqq \{(x, L(x)) \mid x \in \mathbb{R}^d\}.$$

When the window $\mathcal{W}$ is chosen carefully, geometric properties of the associated cut-and-project sets are closely linked to number theoretic properties of $L$. There are two standard choices: the *canonical window* is defined as $[0,1]^k$ and the *cubical window* is defined as $\{0\}^d \times [0,1]^{k-d}$ (both look rather 'cubical', but usually one instead considers projections of these subsets to an 'internal space' complementary to $E$, in which case the canonical window can be a more general polytope). In either case, for $s \in \mathbb{R}^k$ with $\mathbb{Z}^k + s$ not intersecting the boundary of the strip (such points are called *regular*[1]) the resulting cut-and-project sets are 'repetitive' (see Definition 1.2.4), and are not periodic if and only if $L(n) \notin \mathbb{Z}^{k-d}$ for all non-zero lattice points $n \in \mathbb{Z}^d$. Saying that such a cut-and-project set $Y$ is not periodic is to say that $Y + x \neq Y$ for any non-zero $x \in E$.

**Exercise 1.1.1.** Prove the above, i.e., that a canonical or cubical cut-and-project set associated to $L$ will have a period if and only if there is some non-zero $n \in \mathbb{Z}^d$ with $L(n) \in \mathbb{Z}^{k-d}$. Of course there is nothing special about the canonical or cubical window here; you may also like to consider more general 'sufficiently nice' windows (so you should also figure out what 'sufficiently nice' could mean, see also the below exercise).

**Exercise 1.1.2.** Consider a cut-and-project scheme associated to $L$ with $L(n) \notin \mathbb{Z}^{k-d}$ for all non-zero $n \in \mathbb{Z}^d$. Find simple conditions for the window $\mathcal{W}$ which ensure that the associated regular cut-and-project sets are Delone sets (see Definition 1.2.1).

Let us see how Sturmian sequences can be defined via the cut-and-project method. The irrational $\theta$ defines a one-dimensional subspace $E$ of $\mathbb{R}^2$, the line of slope $\theta$ (you may also consider $E$ as the graph of the linear form

[1]One needs to be slightly careful on the definition of the cut-and-project sets associated to non-regular $s \in \mathbb{R}^k$. They should be given as limits of regular cut-and-project sets, more on this in the lectures.

$L(x) = \theta x$). Consider the lifted cut-and-project set $\tilde{Y}$ associated to this setup, with $s = 0$ and cubical window. For each $k \in \mathbb{N}$, there is precisely one $n_k \in \mathbb{N}$ with $(k, n_k) \in \mathcal{S}$, in fact it is easy to see that $n_k = \lceil k\theta \rceil$. Since $k\theta$ is irrational we thus have that

$$n_{k+1} - n_k = \lceil (k+1)\theta \rceil - \lceil k\theta \rceil = (\lfloor (k+1)\theta \rfloor + 1) - (\lfloor k\theta \rfloor + 1) = S_k,$$

so the sequence of jumps $(n_k)_{k\in\mathbb{N}}$ of the vertical displacement precisely matches that of the Sturmian sequence $(S_n)_{n\in\mathbb{N}}$ associated to $\theta$. To form the cut-and-project set $Y$, we project the points of $\tilde{Y}$ onto the line $E$; for suitable projections, the gaps between successive intervals will be one of two lengths, in a sequence precisely matching $(S_n)_{n\in\mathbb{N}}$. Of course, the cut-and-project set extends infinitely to the left too, and we can extend $(S_n)_{n\in\mathbb{N}}$ to a bi-infinite sequence $(S_n)_{n\in\mathbb{Z}}$ in an obvious way. We need to be careful at the origin: we should take $(0,0) \in \tilde{Y}$ but not $(0,1)$ in this case. This is related to $s = 0$ not being a regular value, as mentioned. For regular $s \in \mathbb{R}^2$, the cut-and-project set will correspond to the gaps between values of a 'shifted' Beatty sequence $B_n \coloneqq \lfloor n\theta + r \rfloor$ for $r \in \mathbb{R}$. The non-regular values correspond to $r$ with $n\theta + r = k \in \mathbb{Z}$ for some $n \in \mathbb{Z}$; in this case, there is a good reason to allow for *two* corresponding Beatty sequences with $B_n = k$ and $B_n = k+1$ to be in the family. In the cut-and-project approach, these occur as 'limits' of regular cut-and-project sets, see Subsection 2.1.

**Exercise 1.1.3.** Relate the bi-infinite Sturmian sequence $(S_n)_{n\in\mathbb{Z}}$ associated to irrational $\theta$ to the cut-and-project set with $s = 0$, physical space $E \subseteq \mathbb{R}^2$ the line of slope $\theta$, and *canonical* window (again, because $s$ is not regular you will want to remove a lattice point of the strip near the origin).

There are several ways that all of this could be generalised. For example, one could consider different lattices to $\mathbb{Z}^k$ in the total space, and one could also consider total spaces which are not Euclidean. For example, one may find a cut-and-project scheme whose patterns correspond to Thue–Morse sequences, which is not possible with a Euclidean cut-and-project scheme.

### 1.1.3 Substitution tilings

Consider the *Fibonacci substitution*

$$\sigma(a) = ab, \quad \sigma(b) = a.$$

This rule can be repeatedly applied to any finite word. For example, starting with the letter $a$:

$$a \mapsto ab \mapsto aba \mapsto abaab \mapsto abaababa \mapsto abaababaabaab \mapsto \cdots$$

We've already seen that last word before... in fact, repeated application of $\sigma$ defines longer and longer words, all of which agree precisely with the Sturmian sequence $(S_n)_{n\in\mathbb{N}}$ associated to the golden ratio!

Given a general symbolic substitution $\sigma$ over a finite alphabet $\mathcal{A}$, one considers the collection of *allowed* bi-infinite words $S \in \mathcal{A}^{\mathbb{Z}}$ for which for any finite sub-word $w$ of $S$, there exists a letter $a \in \mathcal{A}$ and some $n \in \mathbb{N}$ for which $w$ is also a sub-word of $\sigma^n(a)$. Under certain conditions on $\sigma$ such allowed sequences $S$ exist and have several interesting properties: they will be non-periodic, repetitive (in fact, linearly repetitive, see Definition 1.2.4) and have the property that $\sigma(S)$ is also an allowed sequence. In fact, one can always *de-substitute* $S$, that is, find a word $S'$ with $\sigma(S')$ equal to $S$ up to a small shift (we can get equality in the geometric setting, allowing the origin to sit in interior of a tile). This imbues the allowed words with an interesting hierarchical structure, which shall be explained in the lectures.

Just as the more symbolic Sturmian sequences could be generalised to higher dimensions with added geometrical considerations, so too can the idea of a sequence generated by a substitution. The details shall be given in the lectures (see also [1]). Also see http://tilings.math.uni-bielefeld.de/ for a nice catalogue of substitution tilings.

## 1.2 Euclidean patterns

We wish to study 'patterns' or 'decorations' of $\mathbb{R}^d$. Here, by this we shall always mean either a tiling or Delone set:

**Definition 1.2.1.** A *tile* of $\mathbb{R}^d$ is a compact subset of $\mathbb{R}^d$ which is equal to the closure of its interior. A *tiling* of $\mathbb{R}^d$ is a set $T$ of tiles for which $\mathbb{R}^d = \bigcup_{t\in T} t$ and distinct tiles intersect only on their boundaries.

A *Delone set* is a subset $X \subset \mathbb{R}^d$ which is *$R$-relatively dense* and *$r$-uniformly discrete* for some $R, r > 0$ i.e., $\mathbb{R}^d = \bigcup_{x\in X} B(x,R)$ and $B(x,r) \cap B(y,r) = \emptyset$ for distinct $x, y \in X$.

There are various ways in which the above definitions could be generalised. For example, we may want to allow tiles to overlap on more than just their boundaries (see Gummelt's representation of the Penrose tilings). One may also wish to weaken the relatively dense and uniformly discrete condition for Delone sets—for example, one may be interested in the point set $X := \{(x, y) \in \mathbb{Z}^2 \mid \gcd(x, y) = 1\}$ consisting of points which are visible from the origin; this set contains arbitrarily large 'holes' and is thus not relatively dense. One frequently also wishes to add labels to tiles of tilings, or points of a Delone sets.

Whatever the form of our pattern $P$, what these situations have in common is that there is a notion of *"the sub-patch of $P$ at the subset $U \subseteq \mathbb{R}^d$"*; we denote this by $P[U]$. More concretely, when our pattern is defined by a tiling $T$ we define

$$T[U] := \{t \in T \mid t \cap U \neq \emptyset\}.$$

For a Delone set $X$,

$$X[U] := \{x \in X \mid x \in U\}.$$

Generally, we shall refer to a decoration $P[U]$ for a pattern $P$ and bounded $U \subseteq \mathbb{R}^d$ a *finite patch of $P$*.

We let

$$P(r) := \{(x, y) \in \mathbb{R}^d \times \mathbb{R}^d \mid P[B(x, r)] - x = P[B(y, r)] - y\}.$$

So $(x, y) \in P(r)$ means that "*the pattern $P$ agrees locally at $x$ and $y$ to radius $r$*". Notice that each $P(r)$ defines an equivalence relation, and that $P(R) \subseteq P(r)$ for $r \leq R$.

### 1.2.1 Finite local complexity

**Definition 1.2.2.** Say that $P$ has *finite local complexity* (FLC) if, for all $r > 0$, there are only finitely many patches $P[B(x, r)]$ up to translation.

**Exercise 1.2.1.** Show that the following are equivalent for a tiling $T$:

1. $T$ has FLC;
2. for all $r > 0$, there exists a compact subset $K_r \subset \mathbb{R}^d$ such that $K_r$ contains a representative of each equivalence class of $T(r)$;

3. there are only finitely many $n$-coronae up to translation in the tiling for each $n \in \mathbb{N}_0$ (see Subsection 2.5.2);
4. there are only finitely many two-tile patches, up to translation. Here, a two-tile patch is a pair of tiles in $T$ intersecting somewhere on their boundaries.

**Exercise 1.2.2.** Show that the following are equivalent for a Delone set $X$:

1. $X$ has FLC;
2. for all $r > 0$, there exists a compact subset $K_r \subset \mathbb{R}^d$ such that $K_r$ contains a representative of each equivalence class of $X(r)$;
3. the set $X - X \coloneqq \{x - y \mid x, y \in X\}$ is discrete and closed;
4. the set $B(0, 2R) \cap (X - X)$ is finite, where $X$ is $R$-relatively dense.

So FLC patterns have a kind of combinatorial nature, they have only finitely many local configurations which piece together to define the entire pattern. Have in mind something like a tiling of a finite number of polygonal tile types (up to translation), always meeting full-face to full-face, or a Delone set where there are only finitely many displacements between 'close' points.

### 1.2.2 Mutual local derivability

**Definition 1.2.3.** A pattern $P$ is *locally derivable* from $Q$ if there exists $r > 0$ so that, if $Q[B(x,r)]-x = Q[B(y,r)]-y$, then $P[B(x,1)]-x = P[B(y,1)]-y$. If, additionally, $Q$ is locally derivable from $P$, then we say that $P$ and $Q$ are *mutually locally derivable* (MLD).

The value of 1 in $P[B(x,1)] - x = P[B(y,1)]$ is arbitrary: the idea is that a sufficiently large radius of the pattern $Q$ at a point $x$ determines precisely the decoration of $P$ to *some* radius at $x$. Loosely, there is a local rule for redecorating $Q$ to get $P$. This could lose information (e.g., scrubbing the colours of an otherwise periodic tiling); if it is a 'reversible' operation then the two patterns are MLD.

**Example 1.2.1.** The Penrose tilings can be represented via 'Penrose rhombs', as Delone sets of vertices of Penrose rhomb tilings (which are cut-and-project sets), 'kite and dart' tilings, 'Robinson triangle tilings' (which are substitution tilings) or via coverings of overlapping Gummelt decagons. All of these are MLD to each other.

**Exercise 1.2.3.** Show that an FLC tiling is always MLD to an FLC Delone set, and vice versa.

**Exercise 1.2.4.** Show that any arrow tiling is MLD to a chair tiling, and any chair tiling is MLD to an arrow tiling (defined as in the lectures).

### 1.2.3 Repetitivity

The above partially justifies our indifference as to the precise representation of our patterns. What's important for us is not the particular choice of how to represent a pattern, but how features of a pattern repeat.

**Definition 1.2.4.** A pattern $P$ is called *repetitive* (or, more specifically, *$\varphi$-repetitive*) if, for sufficiently large $r > 0$, each ball $B(x, \varphi(r))$ contains a representative of every equivalence class of $P(r)$. We call $P$ *linearly repetitive* if we can choose $\phi(r) \leq Cr$ for some $C > 0$.

So an FLC pattern is repetitive if every finite patch of the pattern can be found within some bounded distance of *any* point of the pattern. The notion of $\varphi$-repetitivity gives a more quantitative measure of order: a pattern which is $\varphi$-repetitive for a 'small' function $\varphi$ is considered to be highly ordered. Linear repetitivity was studied by Lagarias and Pleasants as a property signifying high structural order, they called linearly repetitive patterns *perfectly ordered quasicrystals* [23]. Of course, a pattern is periodic if and only if it is $\varphi$-repetitive for $\varphi$ bounded.

**Exercise 1.2.5.** Show that if $P$ is locally derivable from a pattern $Q$ which is $\varphi$-repetitive, then $P$ is $\varphi'$-repetitive with

$$\varphi'(r) \leq \varphi(r + c_1) + c_2$$

for constants $c_1, c_2 \in \mathbb{R}$. In particular, being linearly repetitive is an MLD invariant.

## 1.3 Example research topics in the dynamics and topology of pattern spaces

In Session 2 we shall define the *pattern space* of a pattern. Before doing so, we shall motivate its definition through a couple of examples of its appearance in aperiodic order and elsewhere. Therefore the following shall be a sketchy overview to whet one's appetite, so please bear with the occasional leaps of faith in this section.

### 1.3.1 Dynamics and diffraction

We know of the existence of aperiodically ordered patterns occurring in materials because of their diffraction. In 1982, Dan Shechtman observed a diffraction pattern of a metallic alloy with five-fold rotational symmetry. No periodic arrangement could exhibit this kind of symmetry, but the material was clearly highly ordered due to its beautiful diffraction pattern of sharp Bragg peaks.

One mathematical idealisation of what it means for a pattern to have this kind of diffraction is for it to have 'pure point diffraction'. I don't have the time to cover the ground between the physical idea and the mathematical formalisation (of which see [10, 24, 5]), but here is an abstract version of the property of interest which is equivalent for a large class of patterns: To our pattern $P$, we shall see how to associate to it a dynamical system $(\Omega_P, \mathbb{R}^d)$. In many cases of interest it will be uniquely ergodic, so comes equipped with a certain invariant measure (this is all related to the existence of 'uniform patch frequencies'). Associated to this topological dynamical system there is another: the *maximal equicontinuous factor* (MEF), onto which the dynamical system $(\Omega_P, \mathbb{R}^d)$ factors. Our pattern has pure point dynamical spectrum (we shall sometimes say *pp. diffraction*) precisely when the factor map to the MEF is almost everywhere one-to-one. In this case, measure theoretically the pattern space is conjugate to a rotation on an Abelian topological group (but topologically this is far from true!).

Patterns coming from the cut-and-project method always have pp. diffraction. The question of which patterns arising from substitutions have pp. diffraction is much more interesting and difficult.

**Example 1.3.1.** The tilings associated to the famous Fibonacci substitution

$$\sigma_{\mathrm{F}} = \begin{cases} a \mapsto ab \\ b \mapsto a \end{cases}$$

can also be described in terms of the cut-and-project method, as we saw, so this system has pp. diffraction. The MEF here is the 2-torus $S^1 \times S^1$. More generally, the MEF of a Euclidean cut-and-project set is a $k$-torus, with $k$ the dimension of the total space.

**Example 1.3.2.** The period doubling system with substitution

$$\sigma_{\mathrm{PD}} = \begin{cases} a \mapsto ab \\ b \mapsto aa \end{cases}$$

has pp. diffraction; its MEF is the dyadic solenoid (the inverse limit of the circle $S^1$ with the $\times 2$ covering map). The Thue–Morse system with substitution

$$\sigma_{\mathrm{TM}} = \begin{cases} a \mapsto ab \\ b \mapsto ba \end{cases}$$

factors 2-to-1 to the period doubling system, so its diffraction is not pure point.

The main conjecture in this area is the Pisot Conjecture:

**Conjecture** (Pisot Conjecture)**.** *The dynamical system associated to a* 1*-dimensional irreducible Pisot substitution has pure point dynamical spectrum.*

This conjecture has many forms; some also add the assumption of unimodular. A quick explanation of the terms here: A substitution is *Pisot* if its inflation constant $\lambda \in \mathbb{R}_{>1}$ is a Pisot number i.e., an algebraic integer for which the other roots of the minimal polynomial $p$ of $\lambda$ lie inside the unit circle of $\mathbb{C}$. It is *irreducible* if the characteristic polynomial of the Abelianisation of the substitution is irreducible (so is equal to $p$). It is *unimodular* if the constant term of $p$ is equal to $\pm 1$.

The property of pp. diffraction is determined by the topological dynamical system $(\Omega_P, \mathbb{R}^d)$. And this dynamical system, up to a linear rescaling, is in fact determined just by the topological space $\Omega_P$ due to rigidity results [22]. So there is good reason to expect (and want!) to replace the irreducible assumption of the Pisot conjecture with topological conditions. For some development of this point of view, see [3].

### 1.3.2 Topological invariants of patterns

We see in the above that it is of interest to understand the topology of the space $\Omega_P$. One typically does this by applying some topological invariant, say $I$. One has natural questions:

1. How does one compute $I(\Omega_P)$?
2. Can $I(\Omega_P)$ be understood in a direct way from the structure of $P$?
3. Can the invariant $I(\Omega_P)$ be used to answer questions related to aperiodic order?

Here are some topological invariants that I know of, and these questions addressed for each:

**Example 1.3.3** (Čech cohomology)**.**

1. The most commonly studied topological invariant of $\Omega_P$ is the Čech cohomology $\check{H}^\bullet(\Omega_P)$. We have methods of computing it for substitution tilings (as we shall see) and cut-and-project sets, although the complexity of the computations increases drastically as the dimensions increase (especially the codimension in the case of cut-and-project sets).
2. Whilst there is probably still more to understand in what rôle the Čech cohomology plays in the structure of a pattern, we already understand quite a lot about it. In degree one it describes shape deformations of $P$ [9] and more generally homeomorphisms of $\Omega_P$ [17]. There is a way of visualising it (over $\mathbb{R}$-coefficients) using *pattern-equivariant forms* [18] and (over general coefficients) using *pattern-equivariant cellular cochains* [26] (or chains [31]).
3. The Čech cohomology may play an important rôle in the Pisot conjecture [3]. Kelly and Sadun have demonstrated interesting connections to discrepancy problems in number theory via the cut-and-project method [21].

**Example 1.3.4** ($K$-theory)**.**

1. For low-dimensional tilings (i.e., $d \leq 3$) the situation for computability is as for Čech cohomology. Little is known in higher dimensions; as yet, there is no known counter example to the isomorphisms:

$$K^0(\Omega_P) \cong \bigoplus_{k \text{ even}} \check{H}^k(\Omega_P) \text{ and } K^1(\Omega_P) \cong \bigoplus_{k \text{ odd}} \check{H}^k(\Omega_P)$$

   which hold for FLC patterns of dimensions $d \leq 3$ (but likely only through an inability to make these computations).

2. As ever, one may visualise $K$-theory via vector bundles; one may probably formalise the idea of 'pattern-equivariant vector bundles' to describe these $K$-groups directly from the pattern, but I'm not sure whether or not this would be useful. There is certainly more to learn!

3. $K$-theory features in the *gap-labelling theorem* [6, 7]. It says, roughly speaking, that gaps in the energy spectrum of a Schödinger operator associated to an aperiodic potential can be labelled by elements of the $K$-groups. So this rather abstract topological invariant, amazingly, is linked with physics. This link is made through a noncommutative geometry approach.

**Example 1.3.5** (Homotopy)**.**

1. As we shall see, the classical homotopy groups are not well-suited to studying $\Omega_P$. But one can define certain diagrams of homotopy groups in terms of the *approximants* (see Subsection 2.5.1) and take limits. At present, the computations seem to be restricted mostly to one-dimensional substitution tilings.

2. At current these invariants only have a rather abstract form with respect to the original patterns.

3. Work of Clark and Hunton shows that these invariants can be used to study whether $\Omega_P$ may be given as a subspace of a surface as a codimension one hyperbolic attractor [8].

### 1.3.3 Cut-and-project sets and number theory

The cut-and-project method provides an interesting connection between aperiodic order and number theory. Here we shall briefly outline two such links, to Diophantine approximation and to discrepancy theory.

Let $L$ be a linear form $L\colon \mathbb{R}^d \to \mathbb{R}$. In Diophantine approximation, such a linear form is called *badly approximable* if there exists some $C > 0$ for which

$$\|L(n)\| \geq \frac{C}{|n|^d}$$

for all non-zero $n \in \mathbb{Z}^d$, where $\|x\|$ denotes the distance to the nearest integer from $x$.

**Example 1.3.6.** A linear form of one variable is given by $L(x) = \theta x$, so we may think of it simply as a single number. It can be shown that such a linear form (with irrational $\theta$) is badly approximable if and only if $\theta$ has bounded entries in its continued fraction expansion.

**Theorem 1.3.1** (Haynes–Koivusalo–W [16])**.** *Let $L$ be a linear form with $L(n) \neq 0$ for all non-zero $n \in \mathbb{Z}^d$, and $P$ be an associated codimension* 1 *canonical cut-and-project set. The following are equivalent:*

1. *$L$ is badly approximable;*
2. *$P$ is linearly repetitive.*

This extends a classical result of Hedlund and Morse. Note that the condition $L(n) \neq 0$ simply rules out periodicity. The above theorem has a generalisation to higher codimensions but one must replace the canonical window with the cubical window (the two choices give MLD patterns in the codimension 1 case, but not in higher codimensions).

**Exercise 1.3.1.** Prove the above claim, that a regular codimension 1 cut-and-project set with cubical window is MLD to the corresponding cut-and-project set with canonical window. Show that in higher codimensions ($k-d > 1$) a cut-and-project set with cubical window is locally derivable from that of the canonical window, but that typically the two are not MLD.

Conditions on the linear form $L$ can induce quite strong restrictions on the frequencies of appearances of patterns in cut-and-project sets, see [15]. One may also use discrepancy theory to study these aperiodic patterns [14]. In the reverse direction, one may use aperiodic order and pattern cohomology to study discrepancy theory. To greatly summarise, there are situations where one may identify the problem of showing that a certain sequence has bounded discrepancy, to the problem of showing that a special cohomology class of $\Omega_P$ of an associated pattern $P$ belongs to a special subgroup of 'asymptotically negligible' cocycles. For more details, see [21].

# Session 2

# Pattern spaces

## 2.1 Pattern uniformity/metric

To a pattern $P$ we shall associate a topological space $\Omega_P$ called the *pattern space*. One should think of it as a moduli space of other patterns which are "locally indistinguishable" from $P$.

Let $X$ be a set of patterns of $\mathbb{R}^d$. For $R, \epsilon > 0$, say that two patterns $P, Q \in X$ are $(R, \epsilon)$-*close* if

$$(P + x)[B(0, R)] = (Q + y)[B(0, R)]$$

for $x, y \in \mathbb{R}^d$ with $|x|, |y| \leq \epsilon$. That is, the patterns $P$ and $Q$ agree to radius $R$ about the origin after shifting each by at most distance $\epsilon$ (for non-FLC patterns, it is better to allow a more general kind of '$\epsilon$-wiggle'; see Appendix A).

This defines a *uniformity* on $X$. If you don't know about uniformities then don't worry. The important point is that there is a notion of closeness between patterns: $P$ and $Q$ are considered to be very close as patterns if they are $(R, \epsilon)$-close for some very large $R$ and very small $\epsilon$. This induces a topology on $X$ in the obvious way (think about how it is done for metric spaces), and even the notion of a sequence $(P_k)_{k \in \mathbb{N}}$ in $X$ being Cauchy: when, for any $R > 0$ and $\epsilon > 0$ there exists some $N \in \mathbb{N}$ for which $P_m$ and $P_n$ are $(R, \epsilon)$-close for any $m, n \geq N$. So we can define what it means for $X$ to be complete i.e., when all Cauchy sequences in $X$ converge to a limit in $X$. When $X$ is not complete, we can complete it in an analogous way to how one completes a metric space; the completion is denoted by $\overline{X}$.

**Exercise 2.1.1.** The above can be converted into a metric, although there are different ways of doing this (so my preference is to stick with the less arbitrary object). However, if you would rather think of the geometry as induced by a metric: show that

$$d(P, Q) \coloneqq \min\{1/\sqrt{2}, \inf\{\epsilon > 0 \mid P, Q \text{ are } (\epsilon^{-1}, \epsilon)\text{-close}\}\}$$

defines a metric $d\colon X \times X \to \mathbb{R}$. So we use the function $R(\epsilon) = \epsilon^{-1}$ to remove the independence of the $R$ variable from the $\epsilon$ variable to define the metric; other choices would work too, but one has to be careful then to make sure that the triangle inequality holds (which is why one needs to cap the metric at 1 in the above metric).

## 2.2 Pattern spaces

**Definition 2.2.1.** Let $P$ be pattern with FLC. The *pattern space* of $P$ is defined as

$$\Omega_P \coloneqq \overline{(P + \mathbb{R}^d)}$$

where $P + \mathbb{R}^d$ is the set of translates of $P$ and the completion is taken as described above.

**Remark.** Note that when $P$ does not have FLC (e.g., for $P$ a 'pinwheel tiling'), the geometry on the orbit $P + \mathbb{R}^d$ is not the right one, and $\Omega_P$ if defined as above is not the right space. See Appendix A.

It turns out that we need not think of $\Omega_P$ as a completion. Say that $Q$ is *locally indistinguishable from* $P$ if every finite patch of $Q$ appears, up to translation, in $P$.

**Theorem 2.2.1.** *If $P$ has FLC, then $\Omega_P$ is a compact, connected space which may be identified with the set of patterns which are locally indistinguishable from $Q$ (with the topology defined as above).*

**Exercise 2.2.1.** Prove the above.

**Exercise 2.2.2.** Note that the definition of $\Omega_P$ still makes sense when $P$ does not have FLC, although then it is morally the wrong definition; see Appendix A. Show that $\Omega_P$, as defined above, is compact if and only if $P$ has FLC.

**Exercise 2.2.3.** Show that if $P$ is repetitive then $P$ is locally indistinguishable from any $Q \in \Omega_P$, and vice versa. It follows that $\Omega_P = \Omega_Q$ for any $Q \in \Omega_P$.

**Example 2.2.1.** Let $P$ be a periodic pattern of $\mathbb{R}^d$. Then $P+\mathbb{R}^d$ is a $d$-torus $\mathbb{T}^d$, which is compact and hence already complete, so $\Omega_P = \mathbb{T}^d$.

**Exercise 2.2.4.** Show that if $P$ is FLC but non-periodic, there exists some $Q \in \Omega_P$ with $Q \neq P + x$ for any $x \in \mathbb{R}^d$.

It follows from the above exercise and Theorem 2.3.1 that the space $\Omega_P$ is not path-connected for FLC but non-periodic $P$. Since $\Omega_P$ is always connected, it follows that it is certainly not a CW complex. So typically $\Omega_P$ is a rather complicated space. For only the most basic examples can one meaningfully sketch what these spaces look like:

**Example 2.2.2.** Let $T$ be a tiling of $\mathbb{R}^1$ of tiles unit intervals, with endpoints on $\mathbb{Z}$, with all tiles to the left of the origin coloured black and all those to the right coloured white. The pattern space $\Omega_T$ contains the periodic tiling $B$ of black tiles and the periodic tiling $W$ of white tiles (with endpoints on $\mathbb{Z}$), which are locally indistinguishable from $T$ (although not vice versa!). So $\Omega_T$ contains two embedded circles $S_B$ and $S_W$ consisting of the translates of $B$ and $W$. The pattern space is connected but has three path components: the circles $S_B$ and $S_W$, and the path component containing $T$, which we think of as a line which spirals in to $S_B$ as the origin is moved to the left (i.e., as we consider tilings $T+x$ as $x \to \infty$) and spirals in to $S_W$ as the origin is moved the right (i.e., as we consider tilings $T-x$ as $x \to \infty$). It kind of looks like an everyone's-favourite-stair-traversing-toy™.

**Exercise 2.2.5.** What does the space $\Omega_T$ look like for $T$ the tiling of unit interval white tiles, except with a single black tile?

**Exercise 2.2.6.** Define a tiling $T$ of $\mathbb{R}^1$ of black and white unit interval tiles so that $\Omega_T$ contains precisely $n$ embedded circles. Define another such tiling $T'$ so that $\Omega_{T'}$ contains a copy of every possible tiling of black and white unit interval tiles.

Locally, a pattern space looks like the product $U \times \Xi$ of an open ball $U \subset \mathbb{R}^d$ and a compact, totally disconnected space $\Xi$. The $U$ component parametrises small translates of tilings, and the space $\Xi$ parametrises different ways of completing a large central patch of a tiling to a full tiling. Loosely, this is totally disconnected because such a choice of full tiling consists of a sequence of discrete choices of larger and larger extensions of the central patch.

**Exercise 2.2.7.** Formally prove the claims of the above paragraph.

Recall that a philosophy of ours here is that we only care about properties of tilings which are MLD invariant:

**Exercise 2.2.8.** Show that if $P$ and $Q$ are MLD then $\Omega_P$ and $\Omega_Q$ are homeomorphic. Show that if $P$ and $Q$ are affine transformations of each other, then $\Omega_P$ and $\Omega_Q$ are homeomorphic.

## 2.3 Dynamics

There is an obvious action of $\mathbb{R}^d$ on the space $\Omega_P$, given by translation. Indeed, recall that any point of $\Omega_P$ may be considered as a pattern in its own right. So a vector $x \in \mathbb{R}^d$ acts on $Q \in \Omega_P$ via

$$x \cdot Q \coloneqq Q + x.$$

This makes $\Omega_P$ with the action of $\mathbb{R}^d$ a dynamical system: that is, $0 \cdot Q = Q$ and $y \cdot (x \cdot Q) = (y + x) \cdot Q$ for all $Q \in \Omega_P$ and $x, y \in \mathbb{R}^d$. For FLC pattern spaces, there is quite a tight link between the dynamics and the topology, because of the following:

**Theorem 2.3.1.** *For an FLC pattern $P$, the path components of $\Omega_P$ correspond to the $\mathbb{R}^d$ orbits of the dynamical system i.e., two patterns of $\Omega_P$ are translates of each other if and only if they belong to the same path component of $\Omega_P$.*

**Theorem 2.3.2.** *An FLC pattern $P$ is repetitive if and only if the $\mathbb{R}^d$-action on $\Omega_P$ is minimal, that is every orbit $Q + \mathbb{R}^d$ is dense for all $Q \in \Omega_P$.*

**Exercise 2.3.1.** Prove the above two theorems.

**Exercise 2.3.2.** Show that if $P$ is FLC, repetitive and non-periodic then so is every other $Q \in \Omega_P$. Hence for all $Q \in \Omega_P$ we have a continuous bijection

$$f\colon \mathbb{R}^d \to Q + \mathbb{R}^d$$

onto the path component of $\Omega_P$ containing $Q$, so each path-component is the continuous bijective image of $\mathbb{R}^d$ (note, however, that the map $f^{-1}$ is certainly not continuous). Show that there are in fact uncountably many orbits/path-components in this repetitive case.

## 2.4 Global topology of pattern spaces

The above explains why the pattern space of a non-periodic but repetitive pattern is so complicated, it consists of uncountably many copies of $\mathbb{R}^d$, all tightly scrunched up together to form the compact space $\Omega_P$. However, the following theorem of Sadun and Williams provides a good picture of what these spaces look like globally [28]:

**Theorem 2.4.1** (Sadun–Williams [28])**.** *The pattern space $\Omega_P$ of an FLC pattern $P$ of $\mathbb{R}^d$ is a fibre bundle with fibre a compact totally disconnected space (which is the Cantor space for $P$ repetitive) and base the $d$-torus $\mathbb{T}^d$.*

**Remark.** For those of you who haven't met fibre bundles before: the above intuitively means that $\Omega_P$ is a 'twisted product' of a totally disconnected space $\Xi$ over $\mathbb{T}^d$. More explicitly, there exists a continuous surjection

$$p\colon \Omega_P \to \mathbb{T}^d$$

such that there is a covering of $\mathbb{T}^d$ with open subsets $U$ (which we may take here as homeomorphic to an open ball in $\mathbb{R}^d$) and homeomorphisms

$$h\colon p^{-1}(U) \to U \times \Xi$$

satisfying $p = \pi_1 \circ h$, with $\pi_1$ the projection $(u, x) \mapsto u$.

**Exercise 2.4.1.** Prove the above theorem for an FLC tiling of $\mathbb{R}^1$ of interval tiles, or an FLC Delone set of $\mathbb{R}^1$.

**Exercise 2.4.2.** Prove the above theorem for an FLC tiling of $\mathbb{R}^d$ of unit hypercubes, tiling in the usual periodic fashion, but perhaps labelled with various colours.

Sadun and Williams' proof amounts to showing that we may always reduce to the above case, by converting a more general pattern to a hypercube tiling through 'shape deformations' and MLD equivalences, neither of which change the topology of the pattern space.

## 2.5 Approximant presentations for pattern spaces

### 2.5.1 Inverse limit presentations

An inverse limit diagram (here[1]) is a diagram

$$(\Gamma_\bullet, f_\bullet) = \Gamma_0 \xleftarrow{f_1} \Gamma_1 \xleftarrow{f_2} \Gamma_2 \xleftarrow{f_3} \Gamma_3 \xleftarrow{f_4} \cdots$$

of topological spaces $\Gamma_i$ and continuous maps $f_{i+1}\colon \Gamma_{i+1} \to \Gamma_i$ for all $i \in \mathbb{N}_0$. Given such a diagram its *inverse limit* $\varprojlim(\Gamma_\bullet, f_\bullet)$ (which we shall sometimes denote by $\Gamma_\infty$) is defined by

$$\Gamma_\infty = \varprojlim(\Gamma_\bullet, f_\bullet) = \{(x_i)_{i\in\mathbb{N}_0} \in \prod_{i\in\mathbb{N}_0} \Gamma_i \mid f_i(x_i) = x_{i-1}\},$$

with topology induced from being a subspace of the product $\prod \Gamma_i$ with the standard product topology. We say that a space homeomorphic to $\Gamma_\infty$ has *inverse limit presentation* $(\Gamma_\bullet, f_\bullet)$. Although there need not be metrics here, it is instructive to think of elements of the inverse limit as 'consistent sequences' of points from the approximants, two of which are 'close' when they are sequences which are 'close' on the approximant spaces $X_i$ for 'large' $i$.

[1] More generally, one may consider inverse limit diagrams to be indexed by some poset $(\mathcal{I}, \leq)$ which is directed i.e., for $\alpha_1$, $\alpha_2 \in \mathcal{I}$ there exists some $\beta$ with $\beta \geq \alpha_1, \alpha_2$. A corresponding diagram consists of spaces $\Gamma_\alpha$ for all $\alpha \in \mathcal{I}$ and continuous maps $f_{\beta\alpha}\colon \Gamma_\beta \to \Gamma_\alpha$ for all $\alpha \leq \beta$ satisfying $f_{\alpha\gamma} = f_{\alpha\beta} \circ f_{\beta\gamma}$ whenever $\alpha \leq \beta \leq \gamma$. So the inverse limit diagrams here are always based on the poset $(\mathbb{N}_0 \leq)$, with $\leq$ the usual ordering on $\mathbb{N}_0$.

Notice that we have canonical continuous maps $\gamma_i \colon \Gamma_\infty \to \Gamma_i$, given by sending $(x_i)_{i\in\mathbb{N}_0}$ to $x_i \in \Gamma_i$. These maps of course satisfy $\gamma_i = f_{i+1} \circ \gamma_{i+1}$. As an aside, the inverse limit above is really just an explicit construction of a space satisfying a universal property for the diagram. Precisely, if there is another space $X$ also with continuous maps $\phi_i \colon X \to \Gamma_i$ satisfying $\phi_i = f_{i+1} \circ \phi_i$, then there is a continuous map $g \colon X \to \Gamma_\infty$ making everything in sight commute: $\phi_i = \gamma_i \circ g$ for all $i \in \mathbb{N}_0$.

### 2.5.2 Gähler approximants

As we have seen, the pattern space $\Omega_P$ of an FLC but non-periodic pattern can be very complicated, and is not a CW complex. However, it turns out that $\Omega_P$ always has an inverse limit presentation in terms of CW complexes with cellular maps between them.

Firstly, we shall restrict to FLC *cellular* tilings $T$. This means that $T$ has a CW decomposition into 0-cells, 1-cells,..., $d$-cells, the closed $d$-cells being the tiles. These cells should form part of the decoration for the tiling, so $T[U]$ and $T[V]$ agree up to translation only when their patches of tiles decorated by the cell decomposition agree up to translation.

**Exercise 2.5.1.** Show that any FLC pattern is MLD to an FLC cellular tiling.

The above exercise shows that no generality is lost, up to MLD equivalence, by considering FLC cellular tilings. Given a tile $t$ of the tiling, denote by $T[t, n]$ its *$n$-corona*, defined inductively as follows. The 0-corona $T[t, 0] \coloneqq \{t\}$, i.e., it is the singleton patch containing just $t$. The $n$-corona $T[t, n]$ is the patch of tiles $T[t, n-1]$ along with any other tiles $t$ intersecting $T[t, n-1]$ non-trivially (so necessarily on the boundaries of these new tiles). In other words, we define the $n$-corona by taking $t$ and then iteratively appending adjacent tiles to our patch $n$ times.

Define a relation $\sim_n'$ on $\mathbb{R}^d$ by setting $x \sim_n' y$ if and only if $x$ and $y$ belong to tiles $t_x$ and $t_y$ so that

$$T[t_x, n] - x = T[t_y, n] - y.$$

This won't be a transitive relation for aperiodic tilings (exercise: why not?), so we take its transitive closure to define an equivalence relation $\sim_n$ on $\mathbb{R}^d$.

The quotient space $\Gamma_n = \mathbb{R}^d / \sim_n$ is called the *level $n$ Gähler complex*. A more visual way of viewing this construction is as follows: define an *$n$-collared patch* of $T$ to be a pair $(t, T[t,n])$ of a tile $t$ with its $n$-corona, taken up to translation equivalence. By FLC there are only finitely many $n$-collared patches. To construct $\Gamma_n$, we take a copy of a tile $t$ for each $n$-collared patch $(t, T[t,n])$, and glue them along their boundaries according to how the $n$-collared patches can meet in $T$. These spaces are *branched $d$-manifolds*; locally they look like $d$-dimensional Euclidean space except at a $(d-1)$-dimensional set of branch points. These branch points can be thought of as occurring when one passes over the boundary of an $n$-collared tile and there are several choices of the next $n$-collared tile to jump to.

**Exercise 2.5.2.** Show that the quotient $q\colon \mathbb{R}^d \to \Gamma_n$ maps the cells of $T$ homeomorphically onto $\Gamma_n$, so $\Gamma_n$ carries an induced CW decomposition making the quotient map $q$ a cellular map.

There are canonical cellular maps $f_n\colon \Gamma_n \to \Gamma_{n-1}$, since elements of $\mathbb{R}^d$ identified under $\sim_n$ are certainly identified under $\sim_{n-1}$. We think of these as 'forgetful maps': a point of $\Gamma_n$ describes the location of a tile at the origin, along with some collaring information, and the forgetful map $f_n$ simply forgets some of this collaring information.

**Theorem 2.5.1.** *For a cellular tiling $T$ with FLC, the inverse limit diagram of Gähler approximants defines an inverse limit presentation for the pattern space $\Omega_P$.*

The idea of the proof is as follows: an element of $\Gamma_n$ defines a patch of a tiling of $\Omega_P$ at the origin, and larger and larger such patches for increasing $n$. An element in the inverse limit describes a *consistent* sequence of patches, in other words a sequence of patches $P_0$, $P_1$, $P_2$,... with each $P_n$ nested as a sub-patch of $P_{n+1}$. So the entire sequence of patches determines a tiling of the pattern space. Two of these sequences are 'close' in the inverse limit if and only if they are 'close' on each $\Gamma_1, \ldots \Gamma_n$ for large $n$, which is to say that large central patches of the two tilings agree up to a small translation, so the corresponding tilings are also close in $\Omega_P$, and vice versa.

**Exercise 2.5.3.** Turn the above argument into a rigorous proof.

### 2.5.3 Barge–Diamond–Hunton–Sadun approximants

There is an alternative construction of approximant presentations due to Barge and Diamond in the 1d case, and then extended by them, Hunton and Sadun for general dimensions. Recall the equivalence relation $P(r)$ defined in Section 1.2: $x$ and $y$ are related under $P(r)$ precisely when the '$r$-patch' of tiles within radius $r$ of $x$ agrees with that at $y$ up to a translation from $x$ to $y$. The quotient spaces $K_r := \mathbb{R}^d/P(r)$ are the *BDHS approximants* [4]. For $r \leq R$, the equivalence relation $P(r)$ identifies more points than $P(R)$, so we have a natural quotient map $f_{r,R}\colon K_R \to K_r$ (again, think of it as 'forgetful': a point $x \in K_R$ determines an $R$-patch at the origin, and $f_{r,R}(x)$ remembers only the information of the $r$-patch at the origin).

**Theorem 2.5.2** ([4])**.** *Let $P$ be an FLC pattern and $r_0 \leq r_1 \leq r_2 \leq \ldots$ an increasing sequence with $r_n \to \infty$. Then the corresponding inverse limit*

$$K_{r_0} \xleftarrow{f_{r_0,r_1}} K_{r_1} \xleftarrow{f_{r_1,r_2}} K_{r_2} \xleftarrow{f_{r_2,r_3}} K_{r_3} \xleftarrow{f_{r_3,r_4}} K_{r_4} \xleftarrow{f_{r_4,r_5}} \cdots$$

*of BDHS approximants is a presentation for $\Omega_P$.*

**Exercise 2.5.4.** Prove the above theorem. It is easier than for the Gähler approximants!

Both the Gähler approximants and the BDHS approximants are not much use in making explicit computations for pattern spaces. The approximants can become more and more complex as we go up the ladder. Their use is more of theoretical value, as we shall see in visualising the Čech cohomology of $\Omega_P$ in terms of *pattern-equivariant cochains*.

Whilst, like the Gähler approximants, the BDHS approximants $K_r$ are clearly much tamer than the space $\Omega_P$, there is not an immediately obvious cell structure on each $K_r$. This could be helped by changing the equivalence relations $P(r)$ a little. The power of these approximants, though, is that they can be simple to describe for $K(r)$ with $r$ small, which is of use in making computations for substitution tilings.

## 2.6 Inverse limit presentations for substitution tiling spaces

For a substitution tiling space $\Omega$ we can find a CW complex $\Gamma$ with self-map $f\colon \Gamma \to \Gamma$ so that

$$\Omega \cong \varprojlim(\Gamma \xleftarrow{f} \Gamma \xleftarrow{f} \Gamma \xleftarrow{f} \Gamma \xleftarrow{f} \cdots).$$

The complex $\Gamma$ is defined in terms of the local combinatorics of the tilings (what tiles are there, how can they meet along a vertex, edge, etc.). The self-map $f$ is defined in terms of the substitution. There are essentially two styles here; the original, pioneered by Anderson and Putnam [1], uses Gähler type approximants (which were actually defined later for higher levels by Gähler). We shall look at a later construction due to Barge and Diamond in the one-dimensional case, and extended to higher dimensions later by them, Hunton and Sadun [4].

Let $\sigma$ be a tiling substitution (with various natural properties to make sure everything works, see [1, 4]). It takes a tile type, inflates by some stretching factor $\lambda > 1$, and substitutes the inflated tile with a patch of tiles of the original size. A point of $K_r$ determines a patch of radius $r$ at the origin. So it determines a patch of radius $\lambda r$ of tiles dilated by $\lambda$, and therefore also a patch of radius $\lambda r$ of substituted tiles. This defines the self-map $f\colon K_r \to K_r$ of the BDHS approximant.

**Theorem 2.6.1** ([4]). *Let $T$ be a substitution tiling of $\sigma$ and $r > 0$. With $f\colon K_r \to K_r$ defined as above,*

$$\Omega_T \cong \varprojlim(K_r \xleftarrow{f} K_r \xleftarrow{f} K_r \xleftarrow{f} K_r \xleftarrow{f} K_r \xleftarrow{f} \cdots).$$

To see why the above theorem should hold, note that an element of the inverse limit defines a tiling in a natural way: the $n$th approximant $K_r$ determines a patch of radius $\lambda^n r$ by applying the substitution $n$ times to the $r$-patch determined by the point of $K_r$. This defines a continuous map from the inverse limit into $\Omega_T$, which one may easily show is surjective. Injectivity follows from an assumption called 'recognisability', which shall be discussed in the lectures.

**Example 2.6.1.** Consider a one-dimensional substitution of interval tiles. For $\epsilon > 0$ less than half the length of the tiles, points of $K_\epsilon$ can be one of three sorts: points corresponding to tilings whose origin is *exactly* $\epsilon$ distance from a vertex; *less than* distance $\epsilon$ from a vertex; or *further than* $\epsilon$ from a vertex. The points of the first sort define 0-cells of $K_\epsilon$, and the second two define 1-cells called *vertex-flaps* and *tile-cells*, respectively. A vertex-flap can be thought of as an interval of size $2\epsilon$, one for each way that two tiles can meet at a vertex. There is a tile-cell, an interval of length $l - 2\epsilon$, for each tile of length $l$.

**Example 2.6.2.** Considering all of the above, convince yourself that for the Fibonacci substitution $K_\epsilon$ is a one-dimensional CW complex which has two 'loops' (note that we have only three two-tile patches in a Fibonacci tiling: $aa$, $ab$, $ba$; a $b$ tile cannot be followed by another). Substitution maps one loop around the other, and the second loop maps once around both. As shall be explained in the next session, this is enough information to compute the Čech cohomology of the Fibonacci tilings.

**Exercise 2.6.1.** What complex $K_\epsilon$ (for $\epsilon$ small) and map do you get for the Thue–Morse substitution ($a \mapsto ab$, $b \mapsto ba$)?

# Session 3

# Pattern cohomology

## 3.1 Čech cohomology

Čech cohomology $\check{H}^*(-)$ is a contravariant functor from the homotopy category of topological spaces to the category of $\mathbb{Z}$-graded Abelian groups[1]. In (slightly) more plain English:

1. for each topological space $X$ and $n \in \mathbb{Z}$ we have an Abelian group $\check{H}^n(X)$;
2. each continuous map $f \colon X \to Y$ determines an *induced map*

$$f^* \colon \check{H}^*(Y) \to \check{H}^*(X)$$

   (the arrow going in the 'opposite' direction is what makes the functor 'contravariant');
3. the induced map $\mathrm{id}^*$ of the identity map $\mathrm{id} \colon X \to X$ is the identity on $\check{H}^*(X)$;
4. for two continuous maps $f \colon X \to Y$ and $g \colon Y \to Z$, we have that $f^* \circ g^* = (g \circ f)^*$;
5. for two homotopic maps $f \simeq g$ we have that $f^* = g^*$.

[1]In fact, there is more structure here provided by a cup product, making $\check{H}^*(X)$ a graded ring. We won't consider the product structure here though.

You've likely met singular cohomology before, which satisfies all of the above, and agrees with cellular cohomology on CW complexes (we shall give a loose overview of cellular cohomology later). Unfortunately singular cohomology is not a very useful invariant for pattern spaces: it essentially only probes one path-component at a time (and even these probes only see something they can't tell apart from $\mathbb{R}^d$) but what's interesting about tiling spaces is how the various path components are weaved together. Čech cohomology does provide useful topological information about pattern spaces, and can be given a very geometric description in terms of pattern-equivariant cochains.

We won't formally define Čech cohomology. Fortunately, all we really need to know about it are the following two facts:

1. Čech cohomology is naturally isomorphic to singular (hence cellular) cohomology $H^*(-)$ on the category of spaces which are CW complexes. 'Natural' here you can think of as meaning that not only Čech and cellular cohomology groups 'agree' on CW complexes, but so do induced maps between CW complexes.
2. For a space $X$ with inverse limit presentation $X \cong \varprojlim(\Gamma_\bullet, f_\bullet)$ of compact Hausdorff spaces, we have that $\check{H}^*(X) \cong \varinjlim(\check{H}^*(\Gamma_\bullet), f^*)$.

Item two above says that "*the Čech cohomology of an inverse limit is the direct limit of Čech cohomology groups.* Since pattern spaces are inverse limits of CW complexes, this means that we can understand their Čech cohomology as direct limits of cellular cohomology groups!

### 3.1.1 Direct limits

A *direct limit diagram* of Abelian groups is a diagram

$$(G_\bullet, f_\bullet) = G_0 \xrightarrow{f_0} G_1 \xrightarrow{f_1} G_2 \xrightarrow{f_2} G_3 \xrightarrow{f_3} G_4 \xrightarrow{f_4} \cdots$$

of Abelian groups $G_i$ and homomorphisms $f_i \colon G_i \to G_{i+1}$ for all $i \in \mathbb{N}_0$. As with our description of inverse limits, there is a generalisation to other shapes of diagrams here that we will not need.

The *direct limit* of such a diagram, as a set, is

$$\coprod_{i \in \mathbb{N}_0} G_i / \sim,$$

the disjoint union of the groups $G_i$, with equivalence relation $\sim$ defined by setting $x \sim y$ if $x \in G_m$ and $y \in G_n$ are eventually mapped to the same element of some $G_N$, that is, if $f_{m,N}(x) = f_{n,N}(y)$ for some $N \in \mathbb{N}_0$, where for $i \leq N$ we define $f_{i,N} \coloneqq f_{N-1} \circ f_{N-2} \circ \cdots \circ f_i$. This is made into a group using the obvious operation: $[x] + [y] = [f_{m,N}(x) + f_{n,N}(y)]$ for any $N \geq m, n$. As with inverse limits, this construction is really just an explicit way of defining a group satisfying a certain universal property with respect to the diagram.

**Exercise 3.1.1.** Show that the direct limit is a well-defined Abelian group.

**Example 3.1.1.** Consider the diagram

$$(\mathbb{Z}, \times 2) = \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\times 2} \cdots$$

where each of the maps is given by the $\times 2$ map $x \mapsto 2x$. The direct limit $\varinjlim(\mathbb{Z}, \times 2)$ is the group of dyadic rationals

$$\mathbb{Z}[1/2] = \{x/2^n \in \mathbb{Q} \mid n \in \mathbb{N}_0\}.$$

with the usual operation of addition inherited from $\mathbb{Q}$.

**Exercise 3.1.2.** Prove the above claim, that $\varinjlim(\mathbb{Z}, \times 2) \cong \mathbb{Z}[1/2]$.

**Exercise 3.1.3.** Show that the diagram

$$G \to G \to G \to G \to G \to G \to \cdots$$

has direct limit $G$ if the maps are eventually isomorphisms.

### 3.1.2 Cellular cohomology

We shall not give the full details of the definition of cellular cohomology (see, for example, Hatcher [13]). However, we shall give enough details in the lectures which to demonstrate the main idea and so that computations can be made on simple examples.

For simplicity, let us work only with finite cell complexes $X$; this means that our space $X$ is a CW complex with only finitely many cells (in particular, it is finite dimensional). Our cells will always be assumed to carry some chosen orientation. The *degree $n$ cellular cochains group $C^n$* is isomorphic to $\mathbb{Z}^{c_n}$, where $c_n$ is the number of $n$-cells of $X$. There are *coboundary maps* $\delta^i$ (which shall be loosely described shortly) which form a cochain complex:

$$0 \to C^0 \xrightarrow{\delta^0} C^1 \xrightarrow{\delta^1} C^2 \xrightarrow{\delta^2} \cdots \xrightarrow{\delta^{n-1}} C^d \to 0,$$

where $d$ is the top dimension of cell in $X$. This means that the composition of two consecutive maps is zero: $\delta^{i+1} \circ \delta^i = 0$, that is, $\operatorname{im}(\delta^i) \subseteq \ker(\delta^{i+1})$. Since everything is Abelian, we may take quotients to define the *degree $n$ cellular cohomology group* $H^n(X) \coloneqq \ker(\delta^{i+1})/\operatorname{im}(\delta^i)$. It turns out that these groups do not depend, up to isomorphism, on the choice of cellular decomposition of $X$.

Very briefly, here is how the coboundary maps work (again, for more details consult a reference such as [13]). A generator of $C^n \cong \mathbb{Z}^{c_n}$ should be thought of as an *indicator cochain* $\psi_c$ which assigns value 1 to the (oriented) cell $c$, and 0 to the others. The degree $(n+1)$-cochain $\delta^n(\psi_c)$ is a linear sum

$$\delta^n(\psi_c) \coloneqq \sum_{c' \succeq c} [c, c'] \cdot \psi_{c'}$$

of indicator cochains of $(n+1)$-cells $c'$ with which $c$ is incident. The integers $[c, c']$ are determined by the relative orientations of $c$ and $c'$. For example, in degree zero, for a vertex $v$ we have that $\delta^0(\psi_v)$ is the sum of indicator cochains of edges whose 'head' is $v$, minus those whose 'tail' is $v$. For degree one, suppose for simplicity that each face (2-cell) $f$ is attached to the 1-skeleton via a homeomorphism. The orientation on $f$ induces an orientation on its bounding circle, which either agrees with the orientation on a given edge $e$, in which case $[e, f] = 1$, or is opposite to it, in which case $[e, f] = -1$.

This defines $\delta^n$ on generators of $C^n$, so we extend it to all of $C^n$ linearly, i.e., by setting $\delta^n(\sum k_i \psi_c) \coloneqq \sum k_i(\delta^n(\psi_c))$.

**Exercise 3.1.4.** Show that any connected CW complex $X$ has $H^0(X) \cong \mathbb{Z}$.

**Exercise 3.1.5.** Show that the wedge of $n$ circles $X = \vee_{i=1}^n S^1$ (i.e., the union of $n$ circles, identified at one point) has $H^1(X) \cong \mathbb{Z}^n$. Every finite 1d CW complex is homotopy equivalent to a wedge of some finite number of circles, so this and the above exercise determine $H^*(X)$ for any finite 1d complex.

## 3.2 Computing tiling cohomology

### 3.2.1 Substitution tilings

As we have seen, for substitution tilings one may describe the tiling space as an inverse limit of a single map on a single complex, with all of the data required determined by the short-range combinatorics of the patches and the substitution map. In principle, this makes the cohomology computable. For one-dimensional tilings, for example, one can view everything symbolically, which is simple to feed into a computer, see the Grout program of Balchin and Rust [2].

In higher dimensions things become complicated very quickly. Cubical substitutions are relatively easy to program, but for more general cellular substitutions there is a question of how one describes the combinatorial information to a computer. I have written a computer program which computes for any cellular substitution, but the format of the input data is currently unmanageable (I hope to change this for two-dimensional substitutions at least). The method is based on [31], providing descriptions of the cohomology in very geometric terms as *pattern-equivariant chains*. These programs, even for one-dimensional substitutions, can only return the direct limit diagram and not compute the direct limit, which in general is an extremely difficult algebraic problem. Sometimes (if one only wishes to know the isomorphism type of the direct limit) one may simplify these calculations by replacing a difficult direct limit computation with a sequence of extension problems of easier direct limits, using stratifications of approximants [4].

### 3.2.2 Cut-and-project tilings

Cohomology is also often computable for tilings coming from the cut-and-project method. One may give an inverse limit presentation of the tiling space as inclusions of $k$-tori (where $k$ is the dimension of the total space) with larger and larger expanses of certain irrationally positioned hyperplanes removed, determined by the various pieces of data (the hyperplanes defining the boundary of the window in the 'almost canonical' polytopal window case, and the physical space $E$). To keep all of this data under control it is beneficial to take a slightly more abstract point of view, looking at a certain group homology of a module determined by all of this data. For those interested, see [12].

## 3.3 Pattern-equivariant cohomology

It's all very well being able to compute $\check{H}^*(\Omega_P)$, but what does it actually mean? It's an abstract topological invariant associated to an abstract space, what does it have to do with our original pattern? It turns out that one may visualise the cohomology directly 'on' the original pattern in terms of *pattern-equivariant cochains*.

This approach was originally pioneered by Kellendonk and Putnam [20, 18] using *pattern-equivariant forms*, so necessarily only applied to real-valued cohomology. For general (constant) Abelian coefficients, we can use *pattern-equivariant cellular cochains*, as observed by Sadun [26] based directly on Gähler's construction.

**Definition 3.3.1.** Let $T$ be a cellular tiling with FLC. So $T$ defines a cellular decomposition of $\mathbb{R}^d$; denote the $n$th cellular cochain group by $C^n$. A cochain $\psi \in C^n$ is called *pattern-equivariant* (PE) if there exists some $r > 0$ so that, whenever the patches of tiles with radius $r$ of two $n$-cells $c_1$ and $c_2$ agree up to a translation of $c_1$ to $c_2$, then $\psi(c_1) = \psi(c_2)$ (note that the orientations of $c_1$ and $c_2$ are comparable via this translation). We let $C^n(T)$ denote the group of PE $n$-cochains.

In other words, a cellular cochain is PE whenever that cochain can only 'see' to some finite radius: its value on any $n$-cell is determined completely by the patch of tiles within some fixed radius of that cell.

**Exercise 3.3.1.** Show that if $\psi$ is PE then so is its coboundary $\delta^n(\psi)$, where $\delta^n$ is the usual degree $n$ cellular coboundary map.

From the above exercise, we have a cochain complex

$$0 \to C^0(T) \xrightarrow{\delta^0} C^1(T) \xrightarrow{\delta^1} C^2(T) \xrightarrow{\delta^2} \cdots \xrightarrow{\delta^{d-1}} C^d(T) \to 0$$

of the cellular PE cochain groups, with the usual cellular coboundary maps between them. The corresponding cohomology $H^*(T)$ is called the *pattern-equivariant cohomology* of $T$.

**Theorem 3.3.1** ([26]). *For a cellular tiling $T$ with FLC, we have a canonical isomorphism*

$$\check{H}^*(\Omega_T) \cong H^*(T).$$

*between the Čech cohomology of the pattern space of $T$ and its PE cohomology.*

The proof of the above is remarkably simple, let's sketch it. We have isomorphisms

$$\check{H}^*(\Omega_T) \cong \check{H}^*(\varprojlim(\Gamma_\bullet, f_\bullet)) \cong \varinjlim(\check{H}^*(\Gamma_\bullet), f_\bullet^*) \cong \varinjlim(H^*(\Gamma_\bullet), f_\bullet^*),$$

the first isomorphism coming from Gähler's inverse limit presentation of the tiling space, and the second two coming from properties of Čech cohomology. This latter direct limit is a direct limit of cellular cohomologies of the approximants $\Gamma_i$ and the induced maps between them. The quotient map $q_i \colon \mathbb{R}^d \to \Gamma_i$ is cellular, so a cellular cochain on $\Gamma_i$ pulls back to a cellular cochain on the tiling, which one may show is in fact PE. Moreover, every PE cochain may be given as such a pull-back of a cellular cochain on a $\Gamma_i$ for sufficiently large $i$.

**Exercise 3.3.2.** Turn the above into a rigorous proof.

**Example 3.3.1.** We have $\check{H}^1(\Omega_T) \cong \mathbb{Z}^2$ for the pattern space of a Fibonacci tiling $T$. It turns out that one may choose as generators, in terms of PE cohomology, cochains $\psi_a$ and $\psi_b$, which assign value 1 to every $a$ tile and zero otherwise, and value 1 to every $b$ tile and zero otherwise, respectively. Every other PE cochain can be uniquely written as a $\mathbb{Z}$-linear combination of these two, up to coboundaries of PE 0-cochains.

In top dimension, we have that $\check{H}^d(\Omega_P)$, in terms of PE cohomology, can be thought of as the group of ways of assigning 'charges' to the tiles of $P$, in a way so that identical charges are given to two regions of the tiling whenever those regions agree to some sufficiently large radius, modulo coboundaries of PE cochains. Taking 'modulo coboundaries of PE cochains' essentially means that one is freely allowed to move charges around the tiling, so long as this is done in a way which only depends locally on what the patch of tiles looks like to some radius. This is easily thought of dually, as PE point charges moved around modulo PE paths. One can extend this idea to other degrees via Poincaré duality, giving very approachable geometric descriptions of the generators of the cohomology for certain examples, such as the Penrose tilings, see figures 3.1 and 3.2 [31].

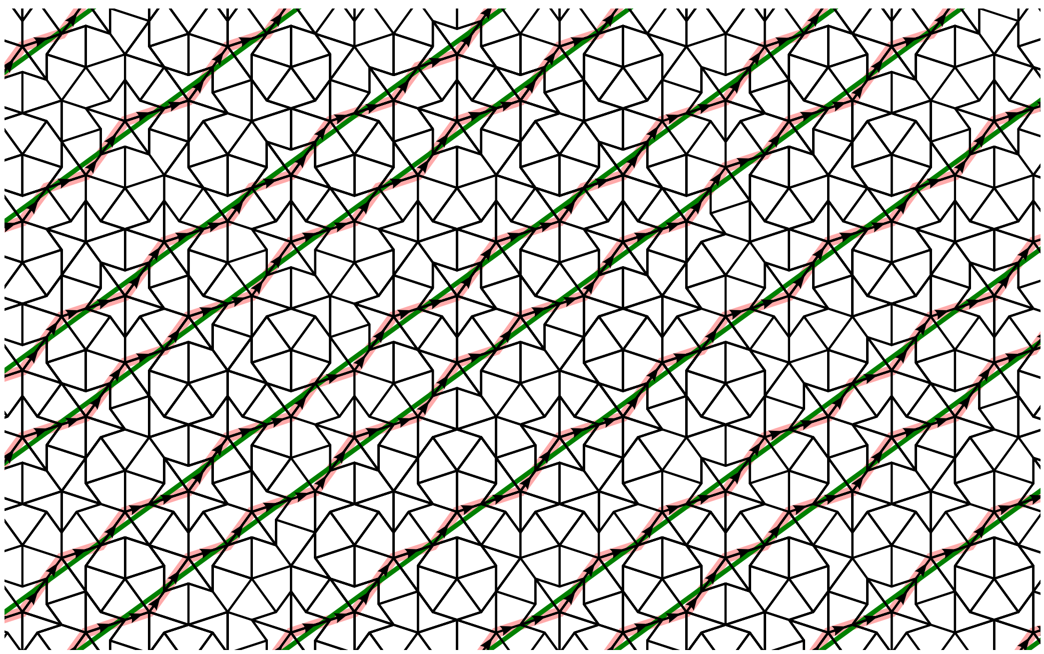

Figure 3.1: PE chain representation of one generator of the degree one cohomology of the pattern space of Penrose tilings, following 'Ammann bars'.

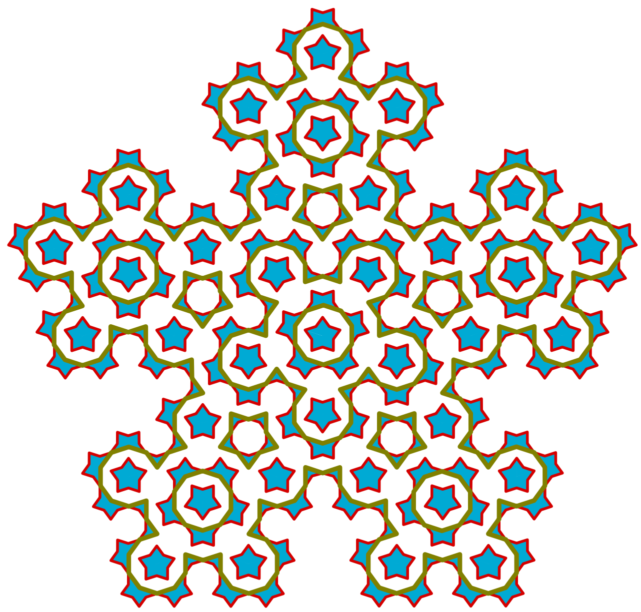

Figure 3.2: PE chain representation of one generator of the degree one cohomology of the pattern space of Penrose tilings, based on loops around the dart tiles.

## 3.4 Further topics

There are many interesting directions in which we could have headed at this point, given the time. There is a natural connection between the degree one cohomology of an FLC pattern and so-called 'shape deformations' of it, see [9] (and more generally, homeomorphisms of $\Omega_P$ [17]). Following the above description of the top degree cohomology group, it is easy to define the *average* of a cohomology class, which defines a group homomorphism $f\colon \check{H}^d(\Omega_P) \to \mathbb{R}$. More generally, in other degrees one has the so-called Ruelle–Sullivan current (e.g. in degree one, very loosely: what is the 'average direction' of a PE 1-cochain?). This is related to the trace map in $K$-theory, and has various applications to the physics of quasicrystals side of the story [19]. There is a notion of *weak* pattern-equivariance, which has nice applications to discrepancy problems in number theory [21]. We have not talked much about patterns which do not have finite local complexity, such as the pinwheel tilings, see [25]. We have also unfortunately had to neglect the rôle of rotations throughout. But interesting rotational symmetry is the reason that we even know about quasicrystals! One may define the pattern space in terms of Euclidean motions of patterns, rather than just translations, which increases the dimensions of the spaces in question. Recent work has determined the cohomology of the Euclidean pattern space of Penrose tilings [30].

# Appendix A

# Tilings of infinite local complexity

When $P$ does not have FLC, the notion of $(R, \epsilon)$-closeness given in Subsection 2.1 is likely not the right one. For example, the Conway–Radin pinwheel tilings has just two tile types up to rigid motion, a $(1, 2\sqrt{5})$ triangle and its reflection, and they meet along boundaries in only a finite number of ways up to rigid motion. However, these triangles can be founded in infinitely many rotational orientations in a pinwheel tiling.

So it is sensible to regard two pinwheel tilings as close if they agree to a large radius about the origin up to a small translation and small rotation. With the notion of $(R, \epsilon)$-closeness from before this change, two otherwise identical pinwheel tilings rotated a tiny amount relative to each other would be counted as distant, which is clearly not quite right.

More generally, we may describe the correct closeness relation for this sort of situation as follows: let $H$ be the space of homeomorphisms of $\mathbb{R}^d$ equipped with the compact-open topology. It is easiest to think of $f_1, f_2 \in H$ as close if $f_1(x)$ and $f_2(x)$ are close for any $x$ which is in some set $K$ containing a large ball at the origin. So a small open neighbourhood of the identity in $H$ consists of homeomorphisms of $H$ which move points at most a small amount unless they are very far from the origin. For $U$ an open neighbourhood of the origin in $H$ and $K$ a bounded subset of $\mathbb{R}^d$, we say that two patterns

$P, Q \in X$ are $(U, K)$-close if

$$H(P)[K] = H(Q)[K].$$

So we think of $P$ and $Q$ as close when they agree to a large radius (parametrised by $K$) up to a small perturbation (parametrised by $U$).

**Exercise 3.0.1.** If you know about uniformities, show that the above defines one.

**Exercise 3.0.2.** Show that if $P$ has FLC, then this new definition does not alter the pattern space $\Omega_P$.

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