Hierarchical tilings and their hulls

Jamie Walton
University of York, UK

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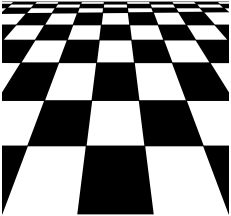
Periodicity

Periodicity is ubiquitous in mathematics. Periodic patterns:

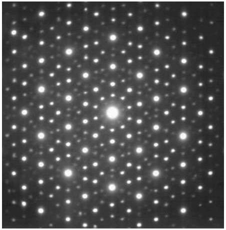
- can be understood globally from a compact fundamental domain;
- are highly ordered;
- can be classified.

The last point is of relevance to crystallographers! There are 230 *space groups* of symmetries of periodic patterns of $\mathbb{R}^3$, which dictate the symmetries of crystals.



Quasicrystals

In the 1980s, Dan Shechtman observed the following diffraction pattern of a rapidly solidified aluminium alloy:



Whatever this substance is, it is:

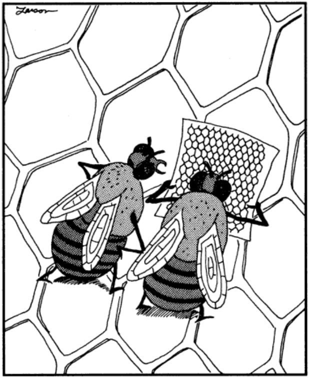
- highly ordered (sharp peaks in diffraction pattern);
- but *not* a crystal! It has fivefold symmetry, forbidden by the crystallographic restriction theorem.

Shechtman's discovery caused a rift in opinion in the crystallography community, and earned him the Nobel Prize in chemistry in 2011.

Aperiodic Order

Aperiodic order aims to study patterns which are highly ordered, but which lack global translational symmetry (periodic patterns are well understood!).

What does it mean for a pattern to be *ordered*?



“Face it, Fred—you’re lost!”

Aperiodic order is, in some ways, a dual theory to fractals:

Fractals:

1. have interesting structure on the **short scale**;
2. often exhibit repetition in structure as one **zooms in**.

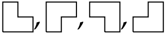
Aperiodically ordered tilings:

1. have interesting structure on the **large scale**;
2. often exhibit repetition in structure as one **zooms out**.

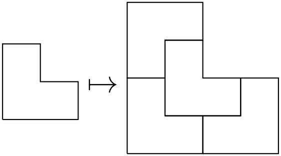
There is a class of tilings which are defined in a way so as to directly inherit such a hierarchy:

Substitution Tilings

A **prototile** P will be a subset of $\mathbb{R}^d$ which is the closure of its interior (usually a polytope).

A **prototile set** $\mathcal{P}$ is a finite collection of prototiles , e.g.,
$\mathcal{P} = \{$$\}$. A **tile** is a translate of a prototile from $\mathcal{P}$. A **patch** is a finite collection of tiles whose interiors are pairwise distinct.

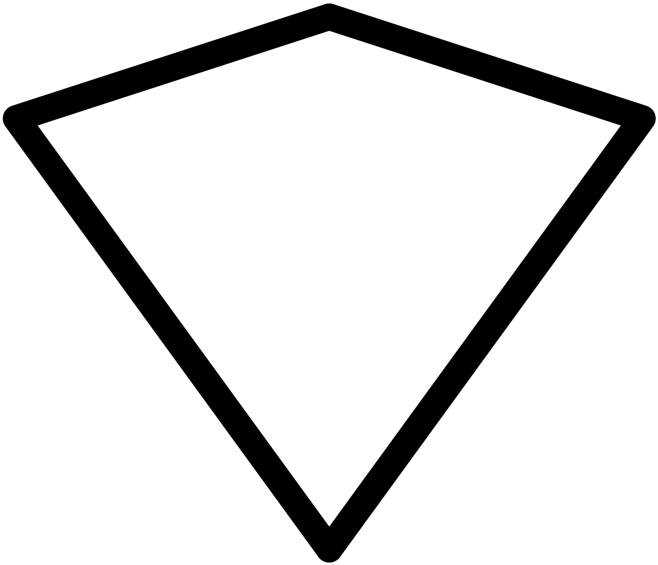
A **substitution rule** ω with **inflation factor** $\lambda > 1$ on a prototile set $\mathcal{P} = \{P_1, \ldots, P_n\}$ assigns to each prototile $P_i \in \mathcal{P}$ a patch $\omega(P_i)$ with support equal to λP_i, e.g.,

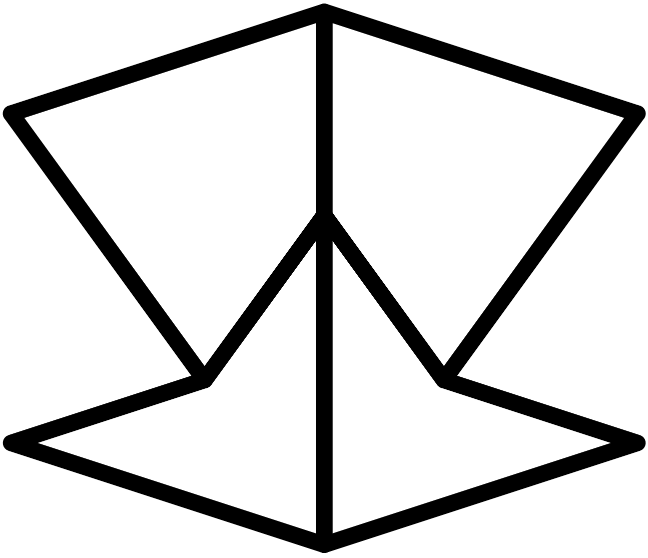


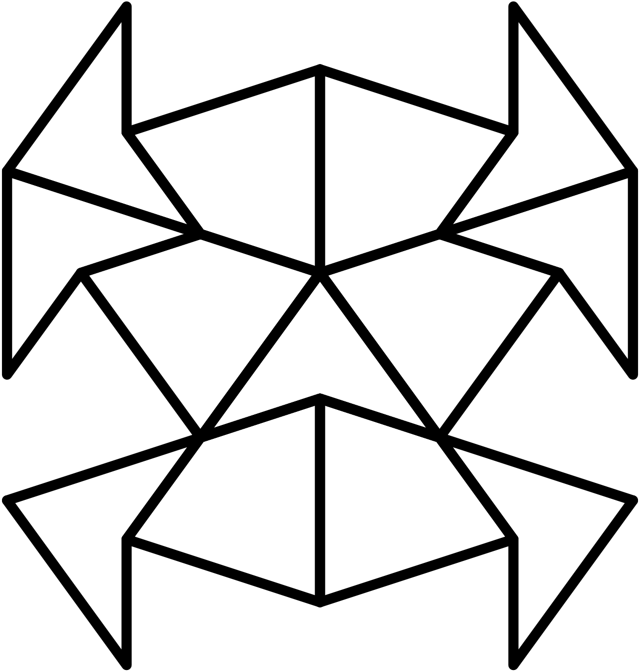
Given a prototile set $\mathcal{P}$ a **tiling** T is a covering of $\mathbb{R}^d$ of tiles whose interiors are pairwise distinct.

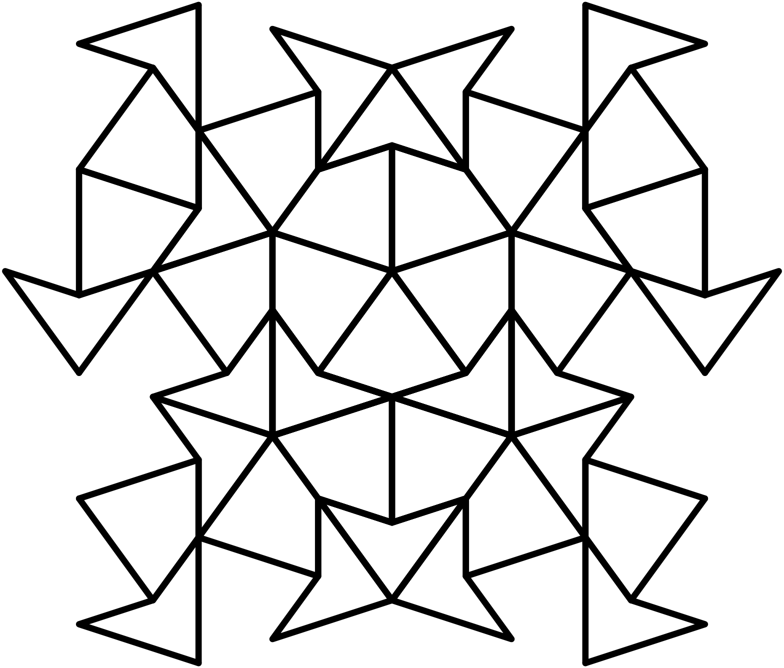
We say that T is **admitted** by the substitution rule ω if, for any finite sub-patch P of T, we have that P is a sub-patch of a translate of some **supertile** $\omega^k(P_i)$.

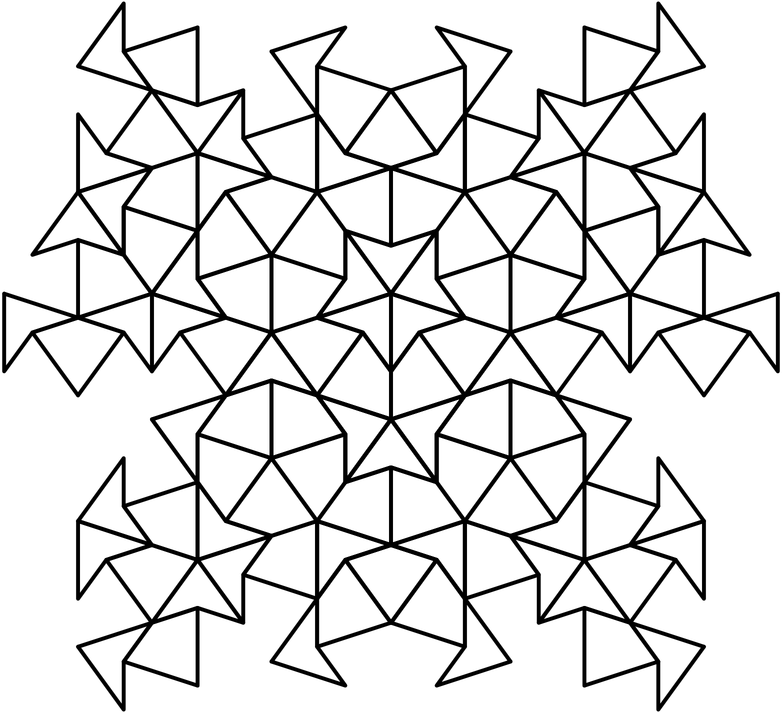
Under relatively weak assumptions on the substitution rule ω, tilings admitted by it exist. Moreover, for any tiling T_0 admitted by ω, there is a **supertiling** T_1 for which the substitution rule takes T_1 to T_0, with T_1 also admitted by ω. This process can be repeated, so we have a hierarchy $T_0, T_1, T_2, \ldots$ of admitted tilings for which $\omega(T_i) = T_{i-1}$.

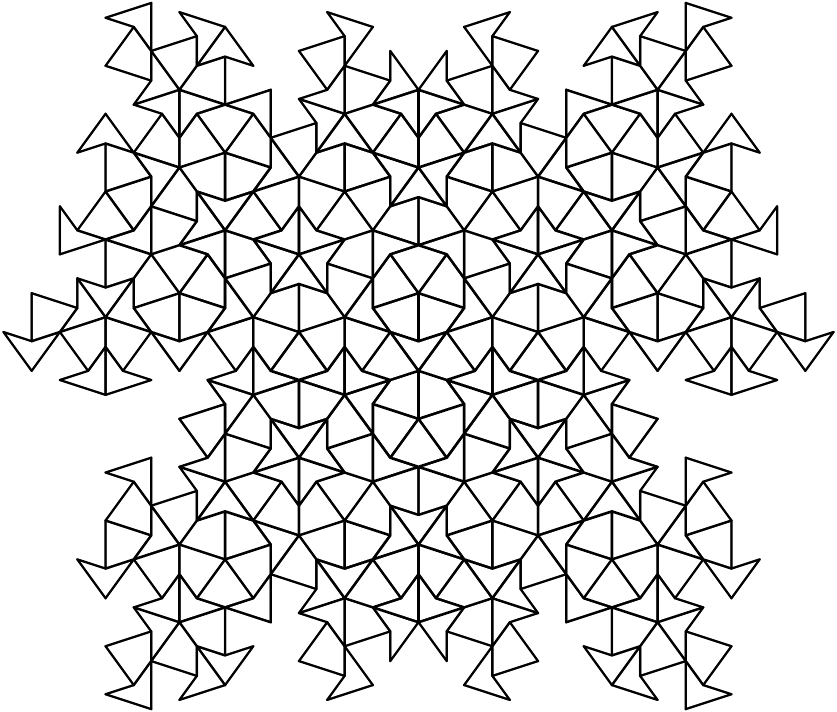


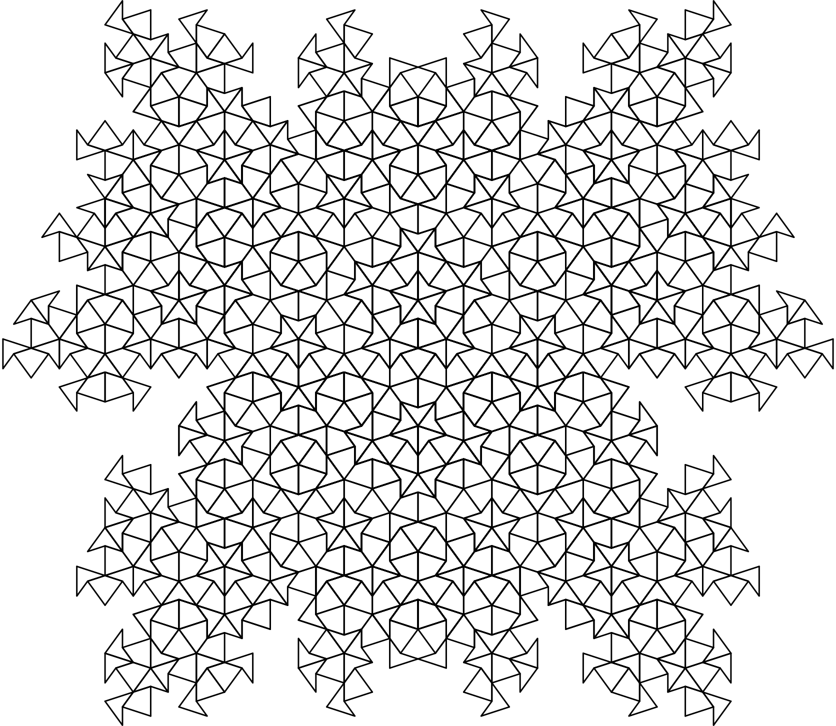


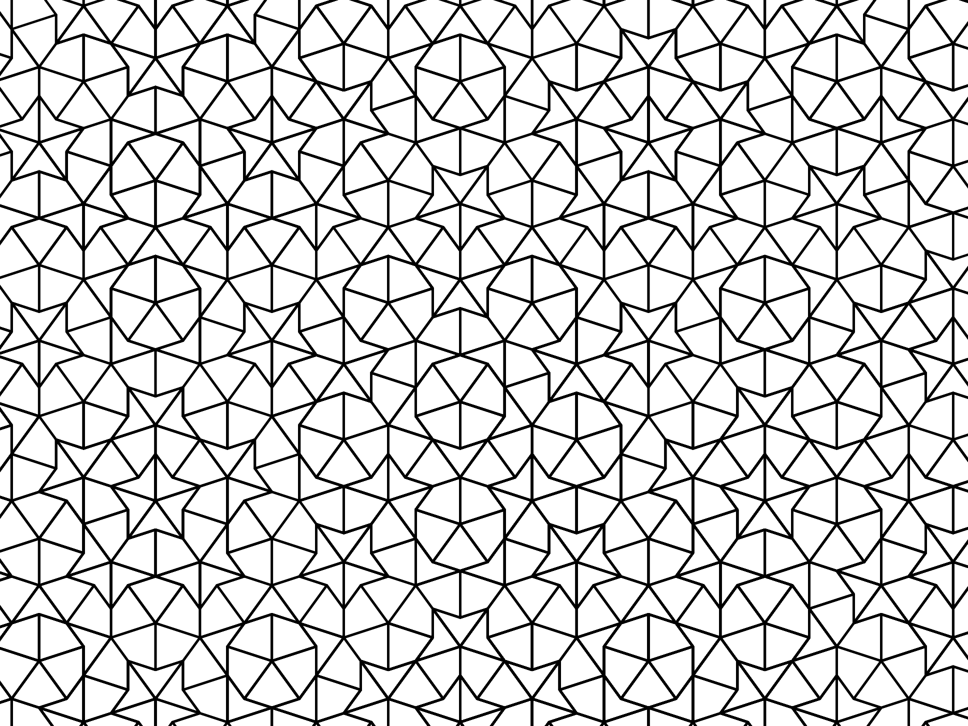


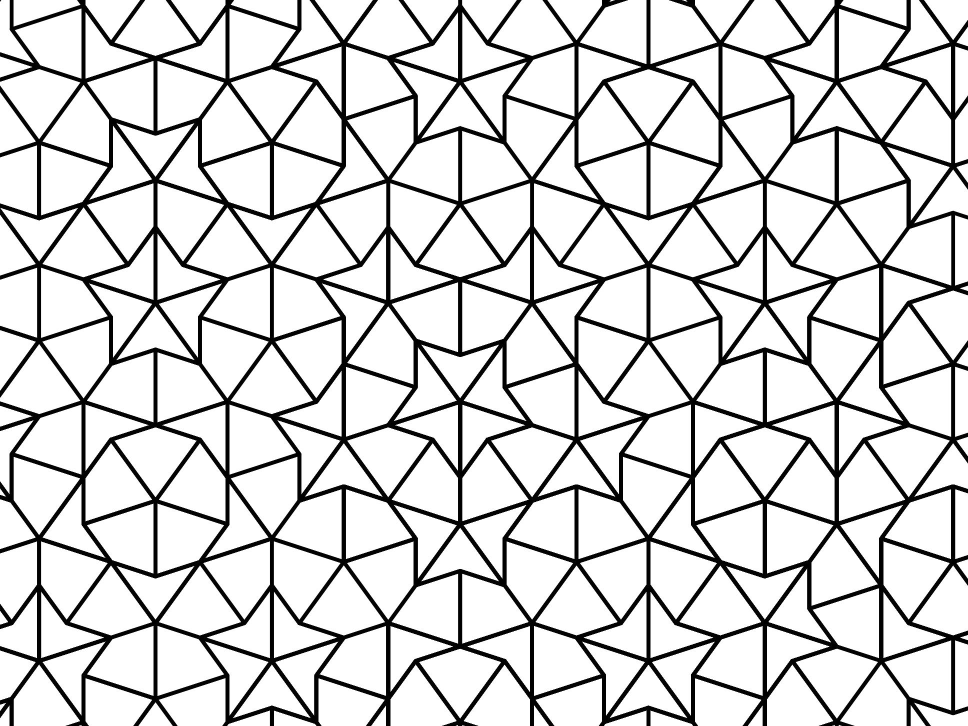


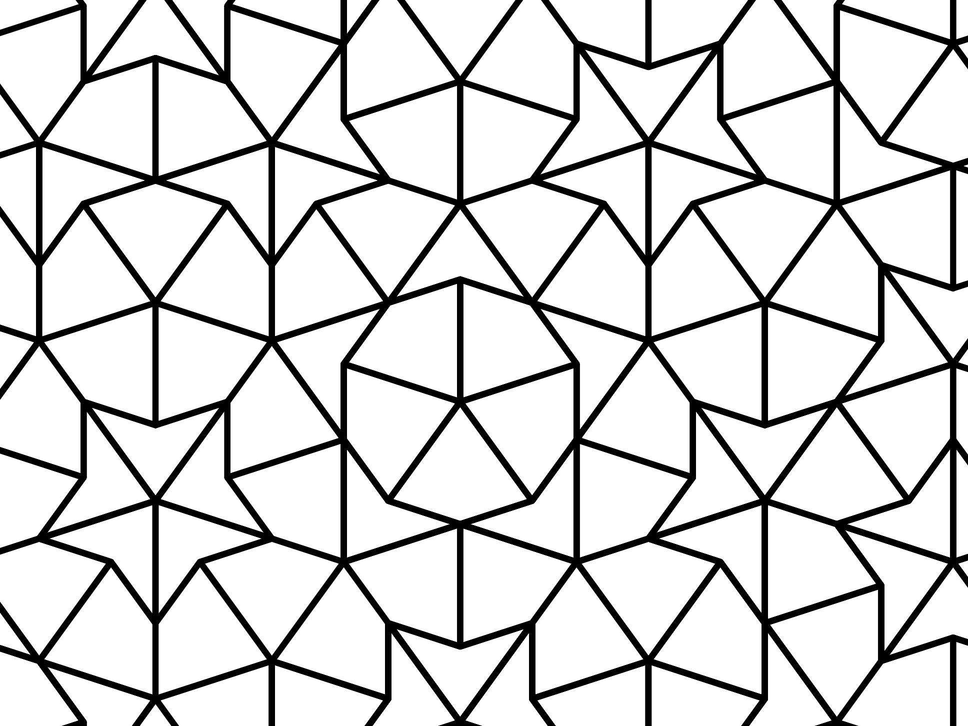


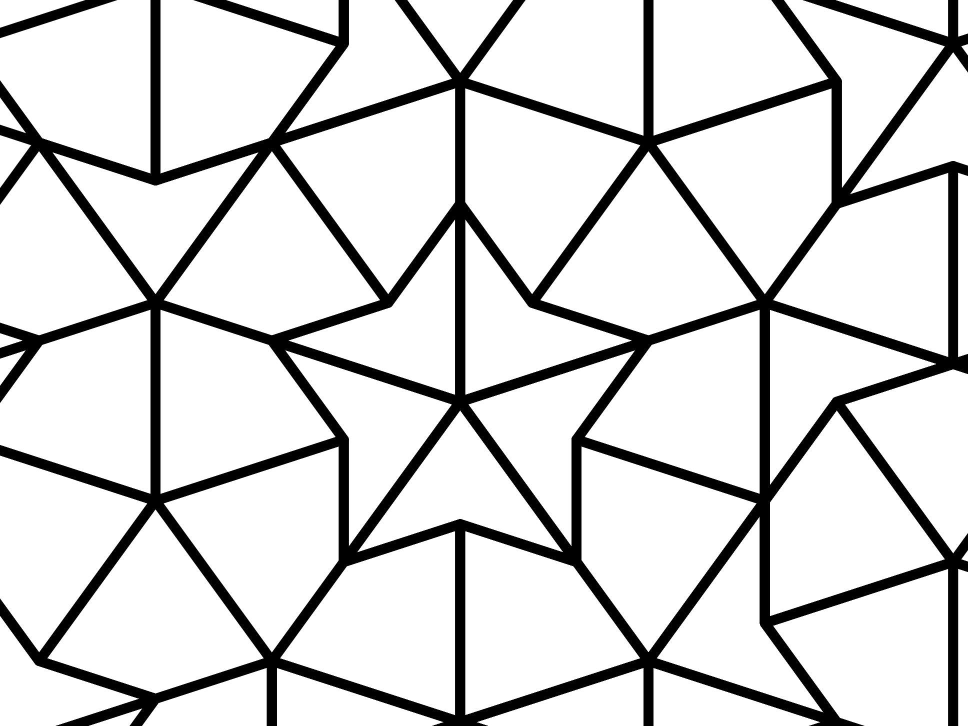


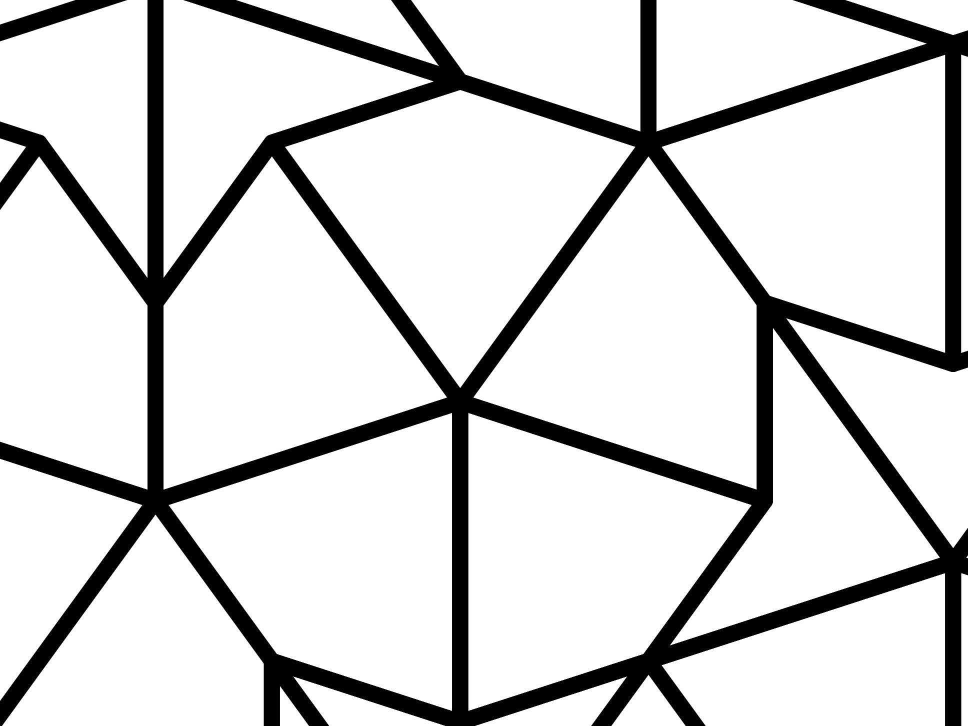


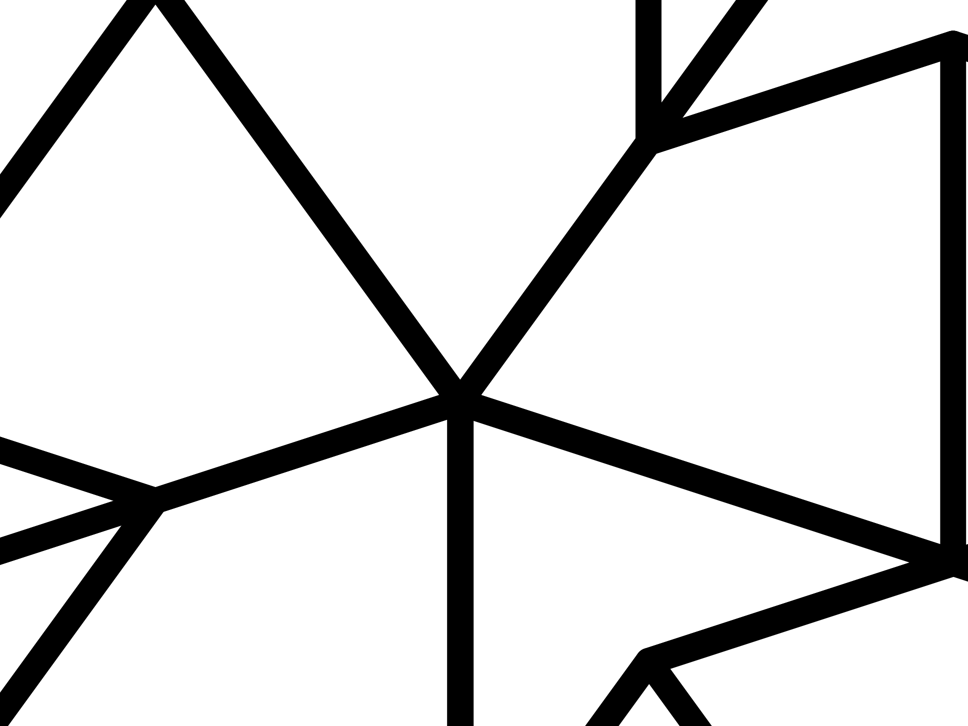


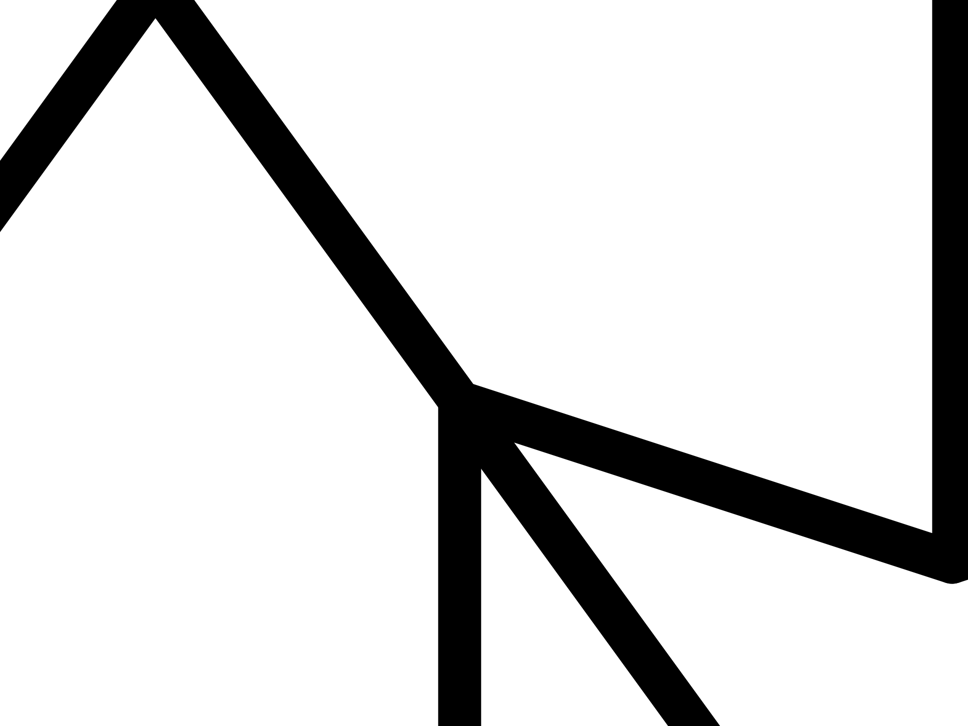












Tiling Spaces

How do we study objects like this? One approach is via the topology, dynamics and ergodic properties of associated moduli spaces of tilings.

We put a geometry (usually a metric or a uniformity) on sets of tilings which, loosely, says:

Two tilings are close if, up to a 'small' perturbation, those tilings agree about the origin to a 'large' radius.

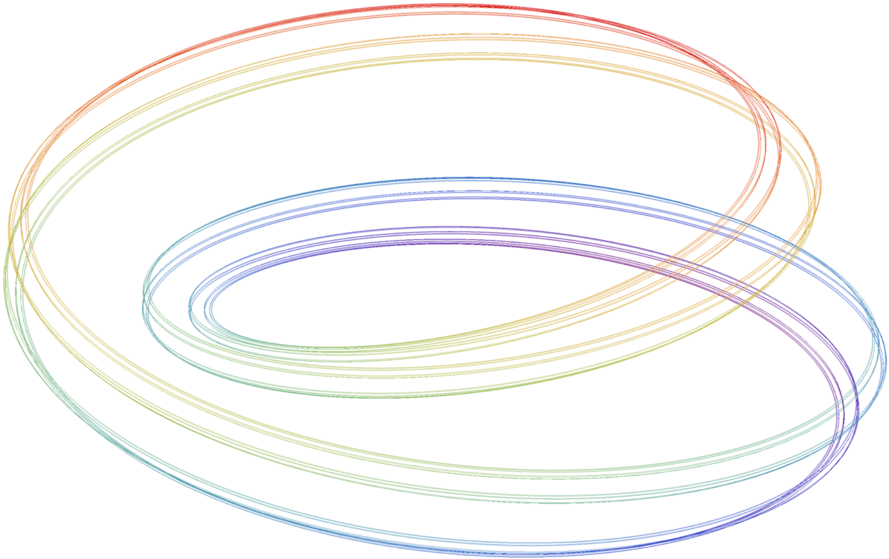
The **translational hull** or **tiling space** Ω of a tiling T of $\mathbb{R}^d$ is defined as

$$\Omega := \overline{T + \mathbb{R}^d},$$

where $T + \mathbb{R}^d$ is the *translational orbit of* T, the collection of tilings given by translations of T.

For nice T (of 'finite local complexity'), Ω is a compact space whose points may be identified with those tilings whose finite patches are translates of the finite patches of T. So Ω is the moduli space of *locally indistinguishable tilings*.

For non-periodic T, this space has complicated topology:



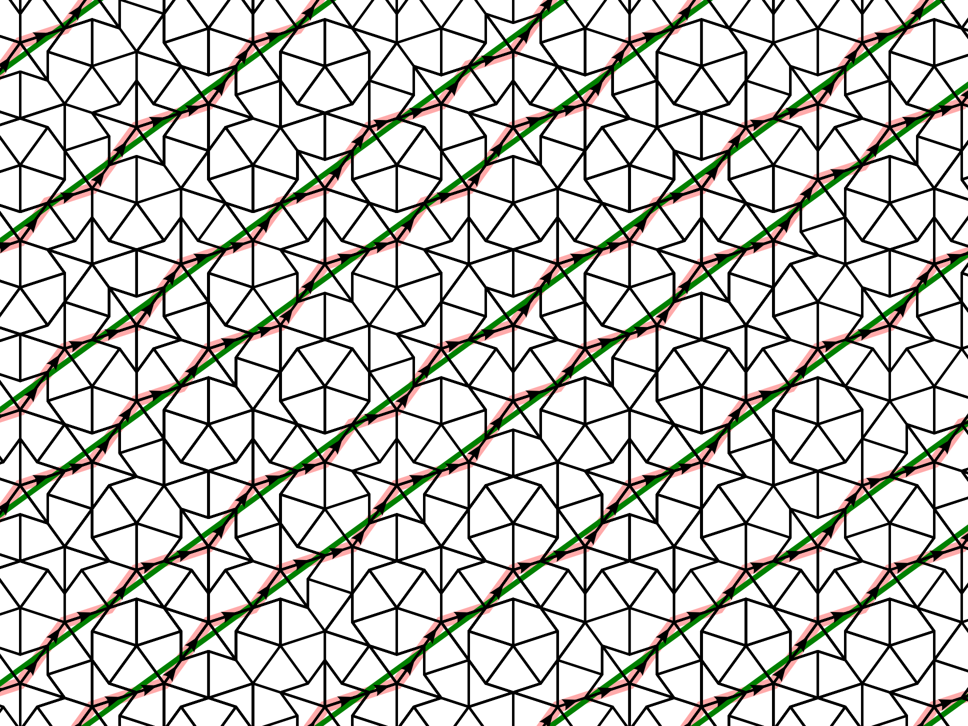
Anderson and Putnam showed that the hull of a substitution tiling may be given as an inverse limit

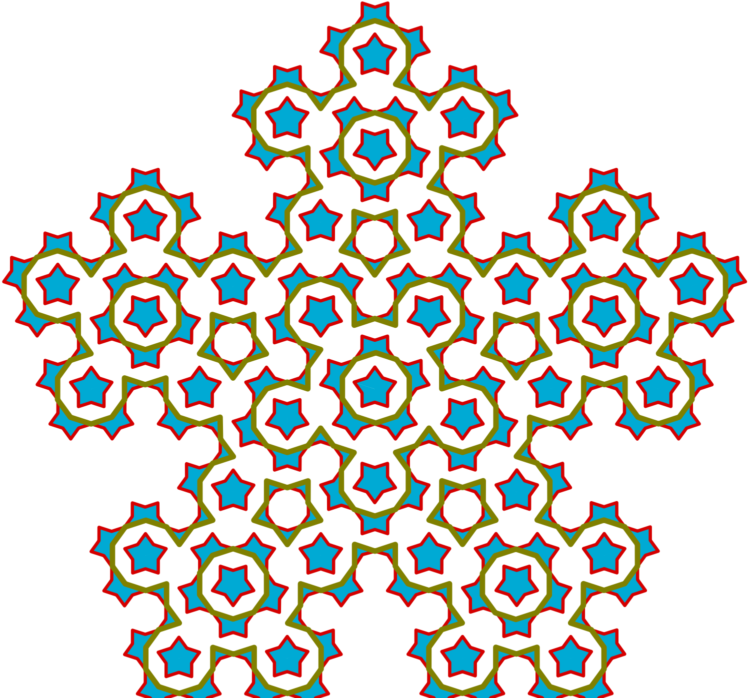
$$\Omega \cong \varprojlim(\Gamma \xleftarrow{f} \Gamma \xleftarrow{f} \Gamma \xleftarrow{f} \cdots).$$

The space Γ is a finite CW complex determined by the short-range combinatorics of the patches which appear in the tilings, and the map f is determined by the action of substitution.

This makes important topological invariants of Ω computable! A commonly studied one is the Čech cohomology $\check{H}^\bullet(\Omega)$.

These cohomology groups have an elegant description, but they are also of principle importance to the structure of tilings. *Pattern-equivariant* descriptions of these groups give digestible and geometric representations of these groups





Thank you!

Picture Credits

- Quasicrystal diffraction image (slide 3) from wikepedia: `https://en.wikipedia.org/wiki/Quasicrystal`
- Penrose rhomb tiling, and wonderful idea of use of Gary Larson's *The Far Side* comic to explain notion of repetitivity (slide 6) from the Tilings Encyclopedia: `http://tilings.math.uni-bielefeld.de/`
- Tiling space (it's actually a solenoid!) image (slide 12) from wikepedia: `https://en.wikipedia.org/wiki/Solenoid_(mathematics)`
- All other images created by author on Inkscape.