Characterising linear repetitivity for polytopal cut and project sets

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Aperiodic Tilings: A Meeting and Mathematical Art Exhibition in Honour of Uwe Grimm at *The Open University*
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Tilings $\rightleftharpoons$ Point Patterns

In Aperiodic Order we are typically interested in *long-range* features, rather than local, more cosmetic choices.

Indeed, one often interchangeably considers patterns which are **mutually locally derivable** (**MLD**).

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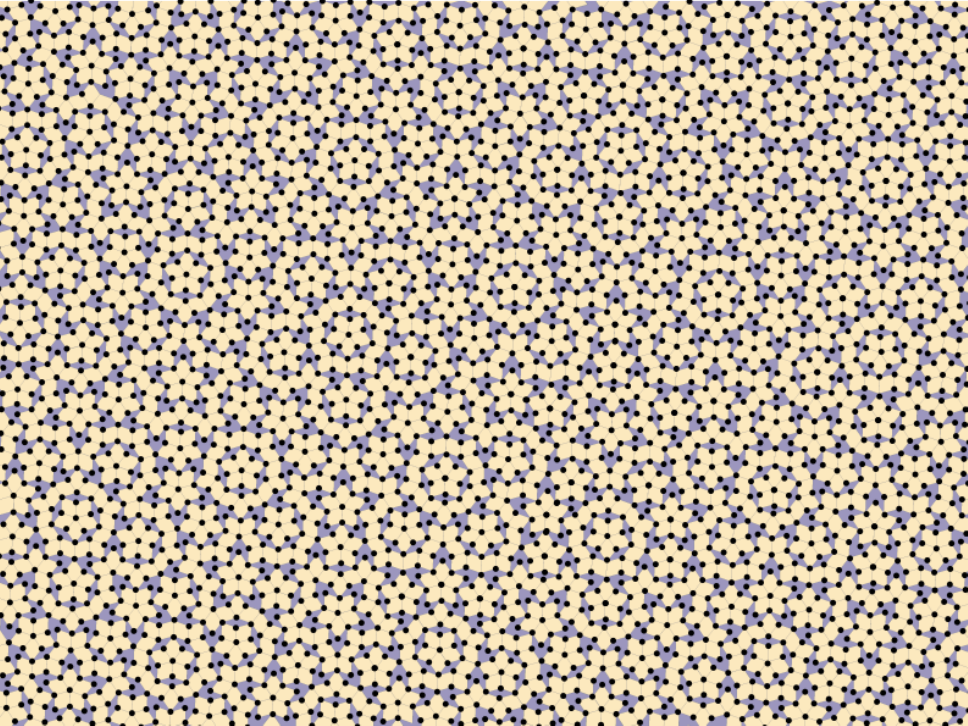
Indeed, one often interchangeably considers patterns which are **mutually locally derivable** (**MLD**).

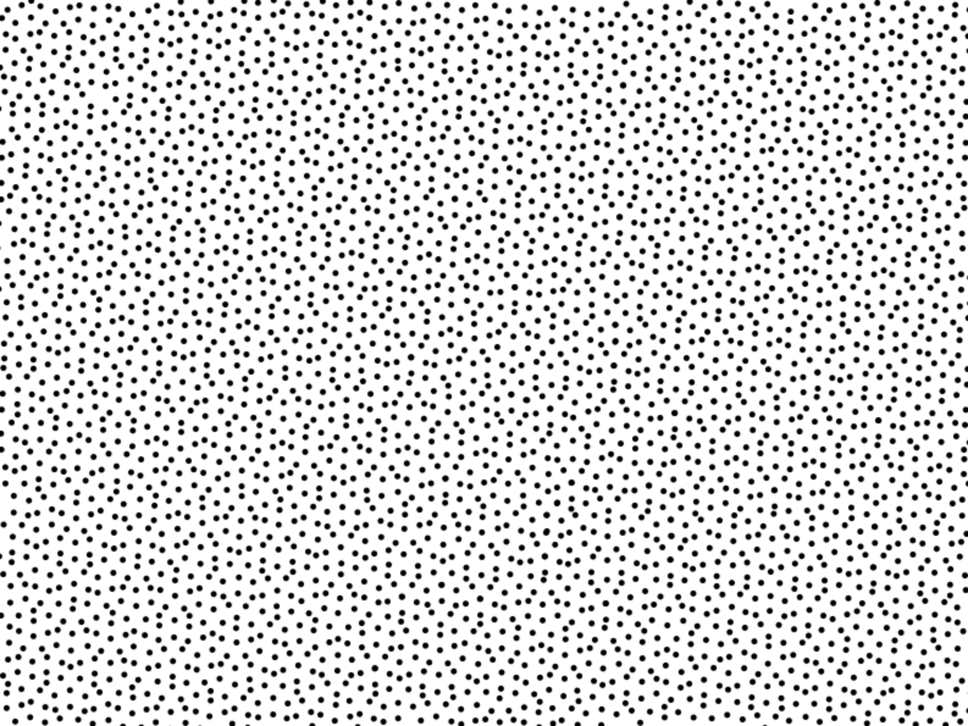
In particular we may interchange:

$$\text{Tilings} \rightleftharpoons \text{Point Patterns}$$

where, roughly,
$\rightharpoonup$ = Marking control points of tiles
$\leftharpoondown$ = Voronoi tessellation

https://tilings.math.uni-bielefeld.de/





Delone Sets

Definition (Delone set)

$\Lambda \subset \mathbb{R}^d$ is a **Delone set** if there exist r, $R > 0$ so that

$$\#\{B_r(x) \cap \Lambda\} \leq 1, \; \#\{B_R(x) \cap \Lambda\} \geq 1 \; \text{ for all } x \in \mathbb{R}^d.$$

i.e., Λ is ‘uniformly discrete’ and ‘relatively dense’.

Many definitions to follow given just for Delone sets but work analogously for tilings.

Complexity and repetitivity

Definition (r-patches)

For $x \in \Lambda$ and $r \geq 0$, we call $P_r(x) := B_r(x) \cap \Lambda$ an r-**patch**. We identify r-patches $P_r(x)$ and $P_r(y)$ if $P_r(x) - x = P_r(y) - y$.

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Definition (Complexity function)

Let $p\colon \mathbb{R}_{\geq 0} \to \mathbb{N} \cup \{\infty\}$ be the **complexity function**:

$$p(r) := \text{number of } r\text{-patches.}$$

Λ has **FLC** (finite local complexity) if $p(r) < \infty$ for all r.

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Definition (Repetitivity function)

For Λ FLC, let $\rho\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ be the **repetitivity** function:

$$\rho(r) = \inf\{R \in \mathbb{R}_{\geq 0} \mid \text{all } r\text{-patches appear in all } R\text{-patches}\}.$$

Λ is **repetitive** if $\rho(r) < \infty$ for all r.

Linear repetitivity

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Linear repetitivity was proposed by Lagarias and Pleasants as a notion of "perfectly ordered quasicrystals". LR is the most repetitive a non-periodic pattern can be, and forces further properties.

Definition (Linear repetitivity)

Λ is **linearly repetitive** if there exists some $C > 0$ so that $\rho(r) \leq Cr$ for all $r \geq 1$ i.e., $\rho(r) \ll r$.

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Linear repetitivity forces the following:

Definition (Low complexity)

We say that Λ has **low complexity** if there exists some $C > 0$ so that $p(r) \leq Cr^d$ for all $r \geq 1$ i.e., $p(r) \ll r^d$.

Cut and project sets. Step 1:

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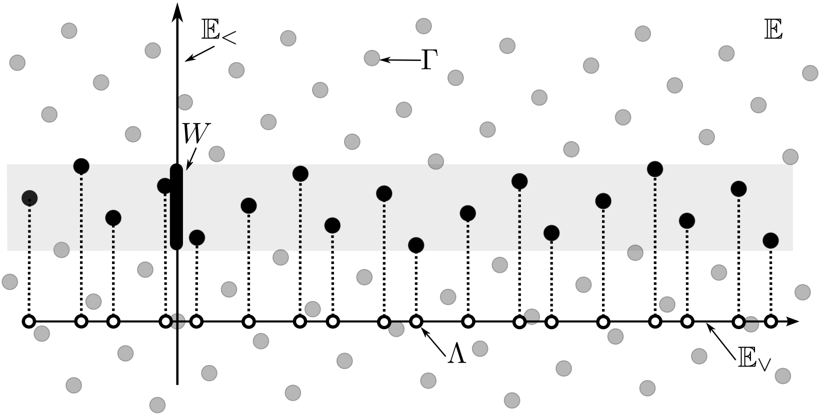

Data of the cut and project scheme:

- The **total space** $\mathbb{E}$
- The **physical space** $\mathbb{E}_\vee < \mathbb{E}$
- The **internal space** $\mathbb{E}_< < \mathbb{E}$
- The **window** $W \subset \mathbb{E}_<$
- The **lattice** $\Gamma < \mathbb{E}$

where $\mathbb{E} \cong \mathbb{R}^k$, $\dim(\mathbb{E}_\vee) = d > 0$, $\dim(\mathbb{E}_<) = n = k - d > 0$, $\mathbb{E} = \mathbb{E}_\vee + \mathbb{E}_<$, W compact with non-empty interior (today: a convex polygon).

We have the projections $\pi_\vee \colon \mathbb{E} \to \mathbb{E}_\vee$ and $\pi_< \colon \mathbb{E} \to \mathbb{E}_<$. We denote $x_\vee := \pi_\vee(x)$ etc.

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We make the following standard assumptions:

1. $\pi_\vee$ is injective on Γ
2. $\pi_<$ is injective on Γ
3. $\Gamma_<$ is dense in $\mathbb{E}_<$

We have the star map $*\colon \Gamma_\vee \to \mathbb{E}_<$ defined by

$$x \mapsto x^* \coloneqq (x^\wedge)_<$$

where $x^\wedge$ is the unique element of Γ with $(x^\wedge)_\vee = x$.

The **cut and project set** is then:

$$\Lambda \coloneqq \{x \in \Gamma_\vee \mid x^* \in W\} \subset \mathbb{E}_\vee$$

i.e., projections of lattice points to the physical space which project to the window in the internal space.

A cut and project scheme defines an infinite family of Delone sets, by translating the lattice, cutting with the ‘strip’ $\mathcal{S} = W + \mathbb{E}_\vee$, then projecting to the physical space.

We only consider **regular** cut and project sets where the (translated) lattice doesn't intersect $\partial\mathcal{S}$ (equivalently, don't project to ∂W), as well as ‘limits’ of the regular patterns at the non-regular lattice translates. Then:

1. these Delone sets are all **locally indistinguishable** from each other (they all have the same finite patches)
2. they're all non-periodic but repetitive

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low complexity:		
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B: In 5 minutes! (for polytopal windows)

C: depends, and is hard. In 1d we have the ‘Pisot Conjecture’

Complexity and repetitivity of polytopal cut and project sets

We will assume that W is a convex polytope, so

$$W = \bigcap_{H^+ \in \mathscr{H}^+} H^+,$$

where $\mathscr{H}^+$ is a finite (and irredundant) set of closed half-spaces in $\mathbb{E}_<$. Associated set of (affine) hyperplanes is denoted $\mathscr{H}$.

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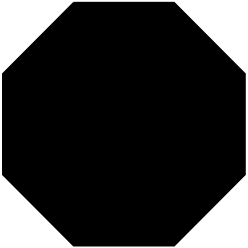
Definition (Weakly homogeneous schemes)

The c&p scheme is called **weakly homogeneous** if there is some 'origin' $o \in \mathbb{E}_<$ so that, for each $H \in \mathscr{H}$, there is some $\gamma \in \Gamma$ and $n \in \mathbb{N}$ with

$$o \in H + \frac{\gamma_<}{n}.$$

Example: Ammann–Beenker

W



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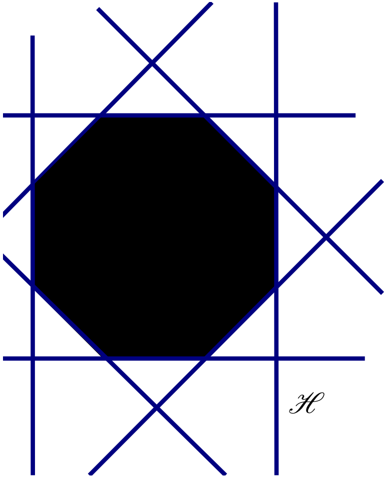

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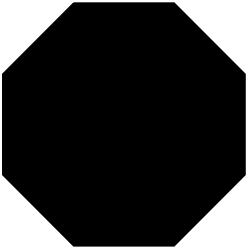

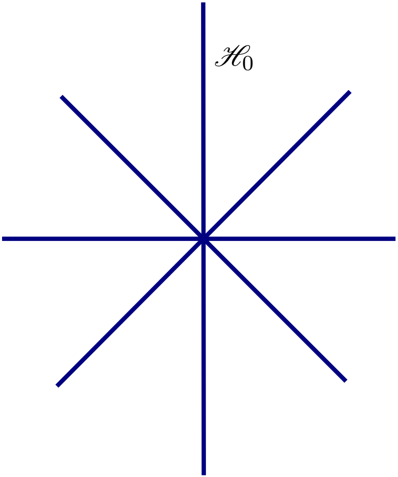

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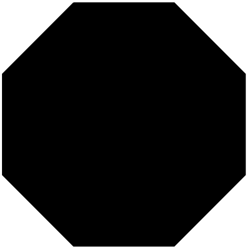

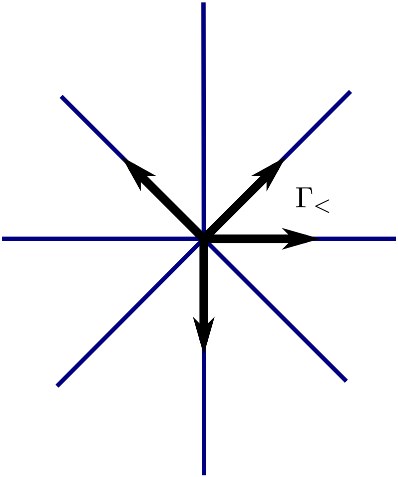

For a polytopal cut and project scheme, the growth rate of $p(r)$ is determined by the pair

$$(\mathscr{H}_0, \Gamma_<),$$

where $\mathscr{H}_0 = \{H - H \mid H \in \mathscr{H}\}$ = the set of codim 1 subspaces of $\mathbb{E}_<$ parallel to the supporting hyperplanes of the window.

Whether the patterns are LR or not is also determined by $(\mathscr{H}_0, \Gamma_<)$ if the scheme is weakly homogeneous.

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Definition (Hyperplane Stabilisers)

For $H \in \mathscr{H}_0$ let

$$\Gamma^H = \{\gamma \in \Gamma \mid \gamma_< \in H\}.$$

Definition (Flags)

A subset f of exactly $n = \dim(\mathbb{E}_<)$ elements in $\mathscr{H}_0$ is called a **flag** if $\bigcap_{H \in f} H = \{0\}$. Let $\mathscr{F}$ denote the set of all flags.

Theorem (Julien 2010, Koivusalo–W 2021)

The complexity function satisfies $p(r) \asymp r^{\alpha}$ *where*

$$\alpha := \max_{f \in \mathscr{F}} \alpha_f, \ \alpha_f := \sum_{H \in f} (d - \mathrm{rk}(\Gamma^H) + \beta_H), \text{ and } \beta_H = \dim \left\langle \Gamma^H_< \right\rangle_{\mathbb{R}}.$$

In short: the number of patches of size r grows polynomially in r, with power $\alpha \in \mathbb{N}$ determined by the ranks of the stabilisers Γ^H and the dimensions of subspaces they span in $\mathbb{E}_<$.

Definition (Diophantine c & p scheme)

Call the scheme **Diophantine** if there exists some $c > 0$ so that, for all non-zero $\gamma \in \Gamma$,

$$\|\gamma_{<}\| \geq c \cdot \|\gamma\|^{-\delta}, \quad \text{where } \delta = \frac{d}{n}$$

(recall: $d = \dim(\mathbb{E}_{\vee})$, $n = \dim(\mathbb{E}_{<})$).

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Example

For $d = n = 1$, up to linear isomorphism,

$$(\Gamma_<, \mathbb{E}_<) \cong (\mathbb{Z} + \alpha\mathbb{Z}, \mathbb{R}),$$

for α irrational. The scheme is Diophantine $\iff$ α is badly approximable $\iff$ continued fraction expansion of α has bounded entries.

We denote:

C $=$ Low complexity
i.e., $p(r) \ll r^d$ for one (equivalently any) c & p set generated by the scheme

D $=$ The scheme is Diophantine

LR $=$ Linear repetitivity
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Notes:

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2. 'indecomposable' means the window isn't a product of lower dimensional polytopal windows (a 'prism'). May be dropped by checking condition for a 'factorisation'
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Class covered includes: canonical cut and project sets, Sturmian sequences, Ammann–Beenker tilings, Penrose tilings*, 'cubical windows', generalising [Haynes–Koivusalo–W 2018]

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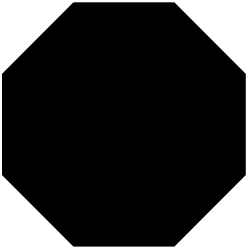

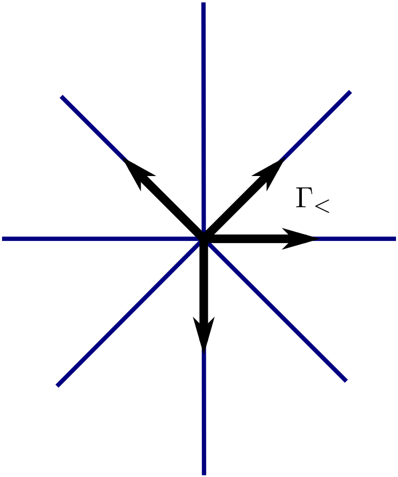

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3. $z + y \notin \Lambda \iff z^* + y^* \notin W \iff z^* \in W^c - y^*$

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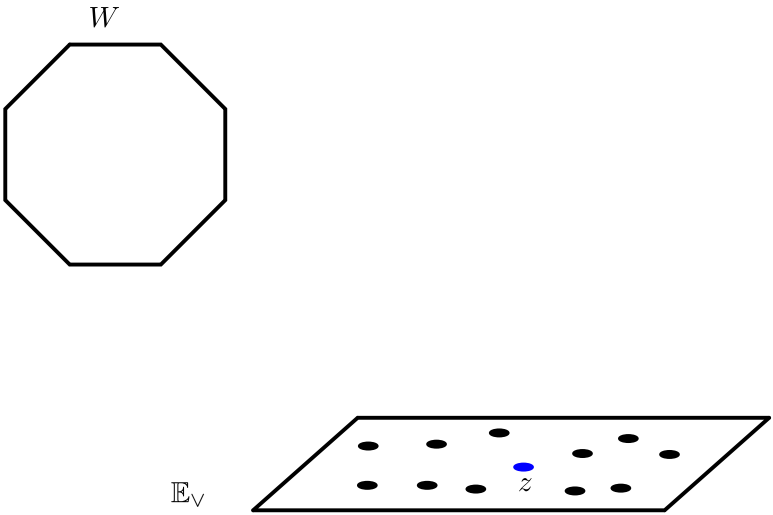
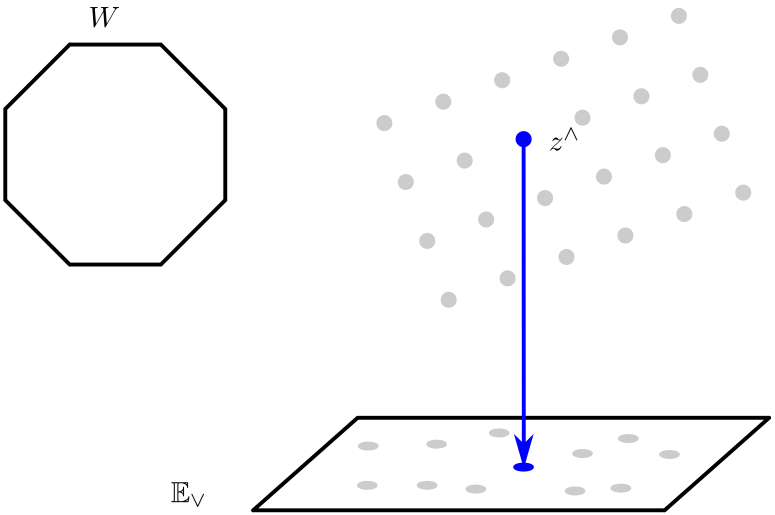
x, y are projections of lattices points a, $b \in \Gamma$. They have relevance to the r-patches only if:

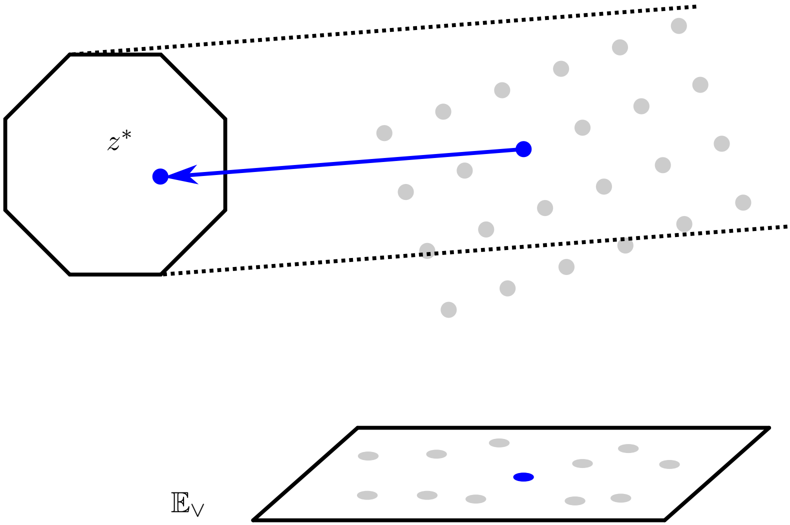
1. $\|a_\vee\|$, $\|b_\vee\| \leq r$
2. $a_<$, $b_< \in W - W$.

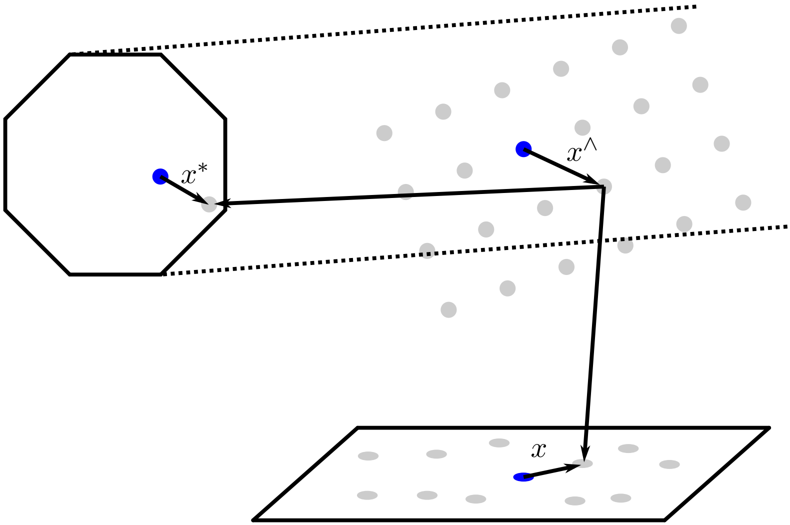
Acceptance domains of r-patches determined as corresponding intersection of $W - x^*$ and $W^c - y^*$ over such lattice points



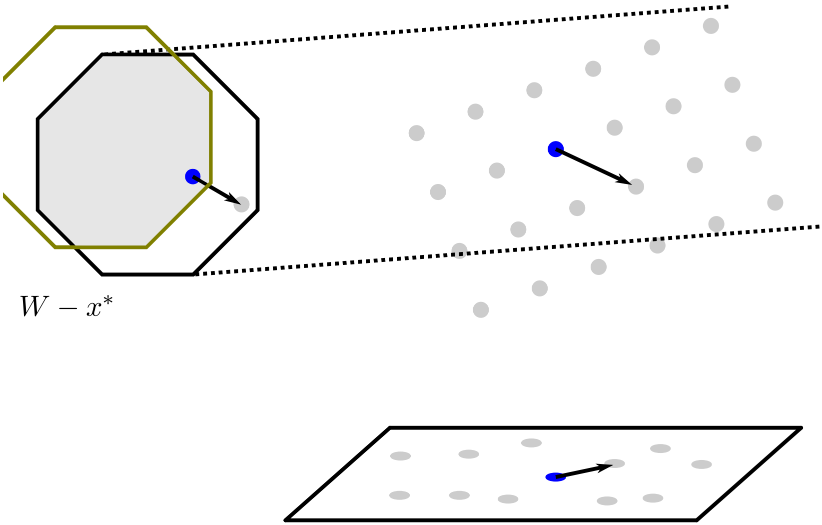
W
$\mathbb{E}_\vee$
z


W
$z^\wedge$
$\mathbb{E}_\vee$

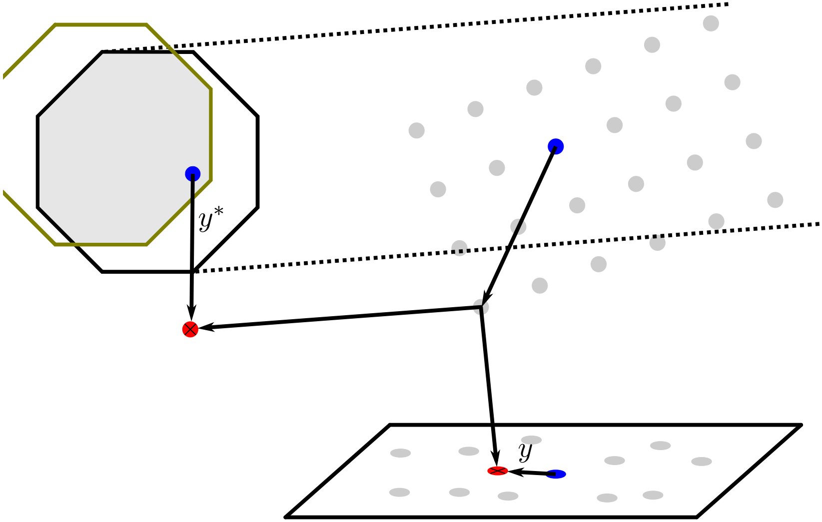

z^*
$\mathbb{E}_\vee$



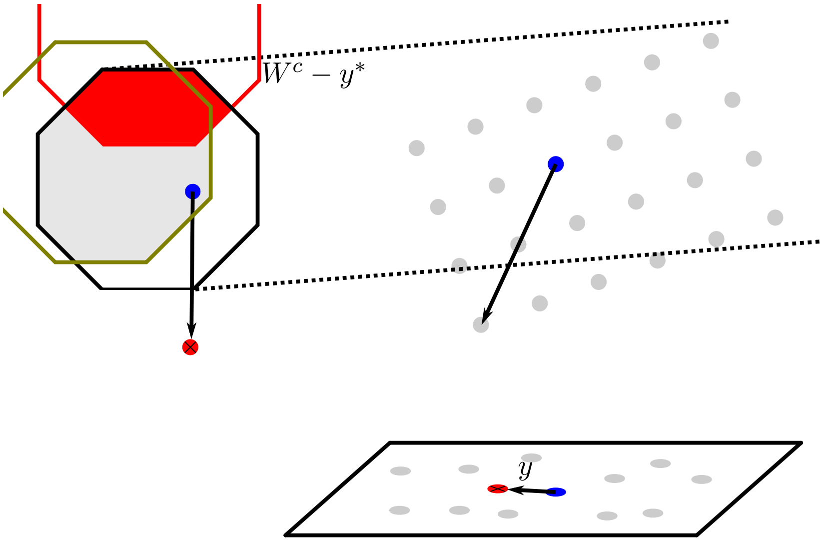
x^*
$x^\wedge$
x

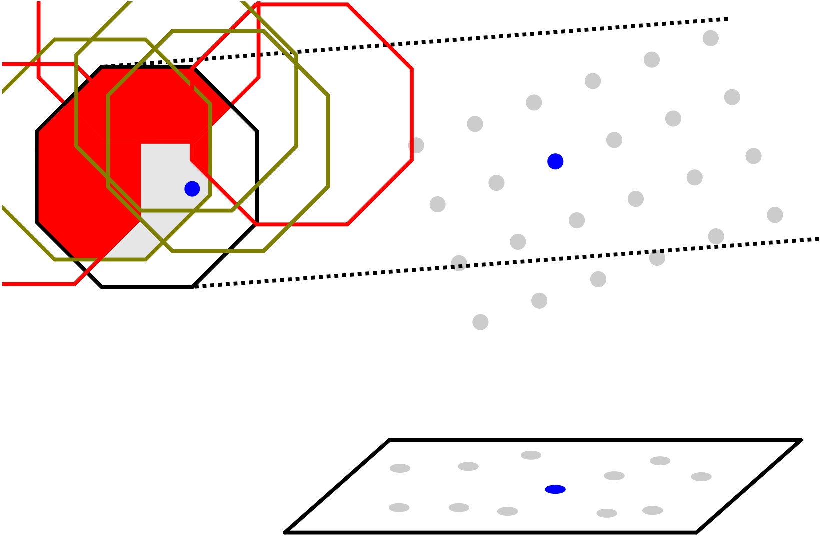


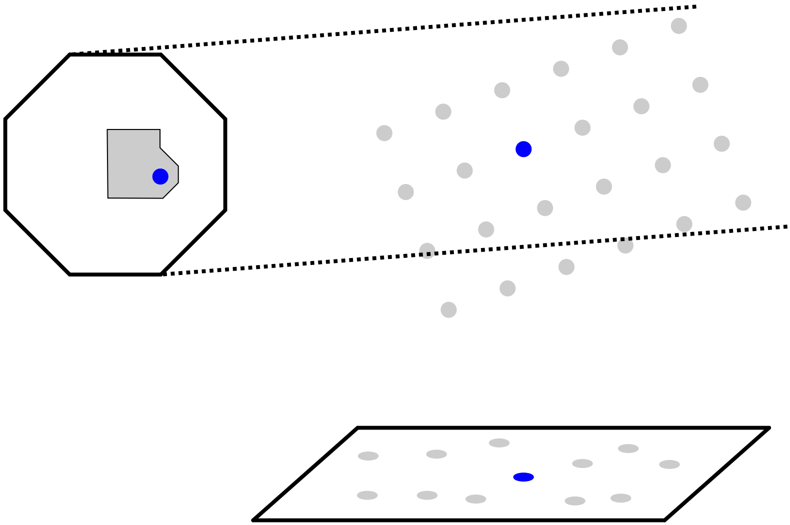
$W - x^*$

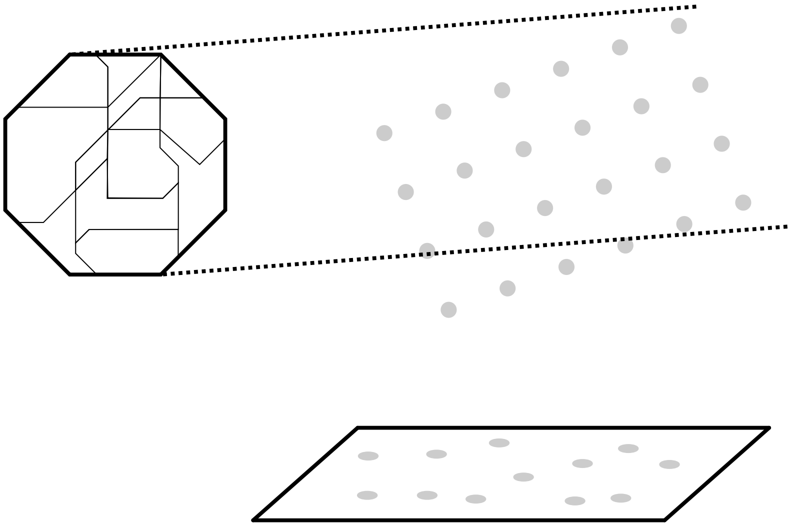


y^*
y


$W^c - y^*$
y







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1. It's easier to count **cut regions** defined by cutting W with $\Gamma_<$ translates of full hyperplanes. Show these cut $\asymp$ same number of regions. These can be counted instead by those 'vertices' $\mathcal{V}_f$ (for flags f, of hyperplanes intersecting to points) inside the window.

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4. Thus, if **D** also holds, vertices stay 'far apart' $\rightsquigarrow$ large acceptance domains.

5. Moreover, **D** $\implies$ r-balls of Γ project to $\mathbb{E}_<$ without large gaps ('transference'). So don't need to travel far from any point of Λ to hit each acceptance domain, since each is 'large' $\rightsquigarrow$ **LR**.

Converse direction is easier: if **C** or **D** fail then we can construct acceptance domains of ‘small volume’ $\leftrightarrow$ patches with ‘low frequency’, so **LR** doesn’t hold.

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- One also has to develop a theory of factors of schemes to deal with the case of decomposable windows.

Some further thoughts / questions:

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Thanks for listening!