Geometric recognisability for FLC patterns

Jamie Walton
University of Nottingham

Automata Seminar, IRIF, Paris
3rd Apr 2026

Overview

1. Symbolic and geometric substitutions
2. Recognisability and main results
3. Filling in the details:
   - generalised patterns
   - (FLC) pattern spaces
   - notion of pattern spaces being 'substitutional' using LD maps ('sliding block codes')

arXiv:2509.21001

Motivation and ideas for some definitions came from collaboration: **When is a cut and project scheme substitutional?**, joint with Edmund Harriss & Henna Koivusalo (arXiv:2512.13659)

Symbolic substitutions (or 'morphisms') on finite alphabets

Example

Fibonacci substitution on alphabet $A = \{a, b\}$

$$\sigma = \begin{cases} a \mapsto ab \\ b \mapsto a \end{cases}$$

Can be iterated:

$a \mapsto ab$

Symbolic substitutions (or ‘morphisms’) on finite alphabets

Example

Fibonacci substitution on alphabet $A = \{a, b\}$

$$\sigma = \begin{cases} a \mapsto ab \\ b \mapsto a \end{cases}$$

Can be iterated:

$a \mapsto ab \mapsto aba$

Symbolic substitutions (or ‘morphisms’) on finite alphabets

Example

Fibonacci substitution on alphabet $A = \{a, b\}$

$$\sigma = \begin{cases} a \mapsto ab \\ b \mapsto a \end{cases}$$

Can be iterated:

$$a \mapsto ab \mapsto aba \mapsto abaab \mapsto abaababa \mapsto abaababaabaab \mapsto \cdots$$

to make infinite or bi-infinite sequences ‘in the limit’:

$$\cdots abaababaabaababaab \cdot abaabaababaabaababaababaaba \cdots \in A^{\mathbb{Z}}$$

Symbolic substitutions (or 'morphisms') on finite alphabets

Example

Fibonacci substitution on alphabet $A = \{a, b\}$

$$\sigma = \begin{cases} a \mapsto ab \\ b \mapsto a \end{cases}$$

Can be iterated:

$$a \mapsto ab \mapsto aba \mapsto abaab \mapsto abaababa \mapsto abaababaabaab \mapsto \cdots$$

to make infinite or bi-infinite sequences 'in the limit':

$$\cdots abaababaabaababaab \cdot abaabaababaabaababaababaaba \cdots \in A^{\mathbb{Z}}$$

More precisely: we take bi-infinite words $w \in A^{\mathbb{Z}}$ s.t. for every factor (finite sub-word) $f \triangleleft w$, $\exists n \in \mathbb{N}$, $\ell \in A$ with $f \triangleleft \sigma^n(\ell)$. This defines a subshift $X_\sigma \subset A^{\mathbb{Z}}$. We may substitute bi-infinite words (letter-by-letter), defining $\sigma \colon X_\sigma \to X_\sigma$.

Hierarchy / 'desubstitution'

$\sigma\colon X_\sigma \to X_\sigma$ is 'essentially surjective': letters can be grouped into 'superword' in X_σ. More precisely: $\forall w \in X_\sigma$, $\exists w' \in X_\sigma$ with $w =$ (some shift of) $\sigma(w')$.

$$\cdots \underbrace{ab}_{a}\underbrace{a}_{b}\underbrace{ab}_{a}\underbrace{ab}_{a}\underbrace{a}_{b}\underbrace{a\cdot b}_{a}\underbrace{a}_{b}\underbrace{ab}_{a}\underbrace{ab}_{a}\underbrace{a}_{b}\underbrace{ab}_{a}\underbrace{ab}_{a}\underbrace{a}_{b}\cdots$$

This grouping is unique for the Fibonacci ('injectivity' of σ).

Hierarchy / ‘desubstitution’

$\sigma\colon X_\sigma \to X_\sigma$ is ‘essentially surjective’: letters can be grouped into ‘superword’ in X_σ. More precisely: $\forall w \in X_\sigma$, $\exists w' \in X_\sigma$ with $w =$ (some shift of) $\sigma(w')$.

$$\cdots \underbrace{ab}_{a}\underbrace{a}_{b}\underbrace{ab}_{a}\underbrace{ab}_{a}\underbrace{a}_{b}\underbrace{a\cdot b}_{a}\underbrace{a}_{b}\underbrace{ab}_{a}\underbrace{ab}_{a}\underbrace{a}_{b}\underbrace{ab}_{a}\underbrace{ab}_{a}\underbrace{a}_{b}\cdots$$

This grouping is unique for the Fibonacci (‘injectivity’ of σ).

Example

Consider the substitution $\sigma(a) = \sigma(b) = ab$ on $A = \{a, b\}$:

$$a \mapsto ab \mapsto abab \mapsto abababab \mapsto \cdots .$$

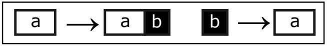
The associated bi-infinite words are periodic, and the grouping is no longer unique:

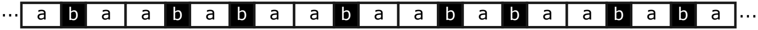
$$\cdots \underbrace{ab}_{a/b?}\underbrace{ab}_{b/a?}\underbrace{ab}_{a/b?}\underbrace{ab}_{b/a?}\underbrace{ab}_{a/b?}\underbrace{ab}_{b/a?}\underbrace{ab}_{a/b?}\underbrace{ab}_{b/a?}\cdots$$

Geometric inflation rules

In the Fibonacci, associate as to an interval 'tile' of length $\varphi = \frac{1+\sqrt{5}}{2}$, and bs with tile of length 1.

We thus define an associated geometric substitution rule, given by: first '**inflate** by φ', then **replace**:

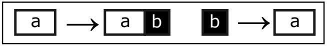


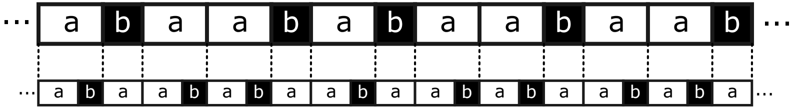


Geometric inflation rules

In the Fibonacci, associate as to an interval 'tile' of length $\varphi = \frac{1+\sqrt{5}}{2}$, and bs with tile of length 1.

We thus define an associated geometric substitution rule, given by: first '**inflate** by φ', then **replace**:

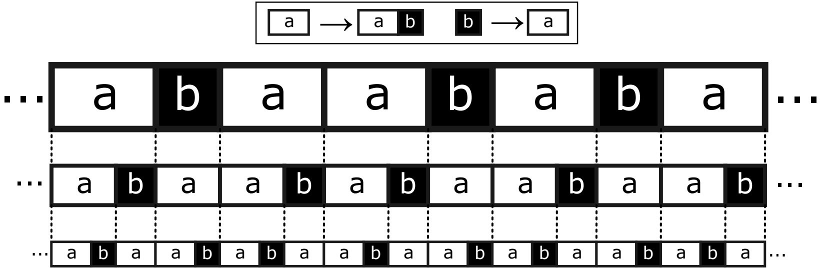




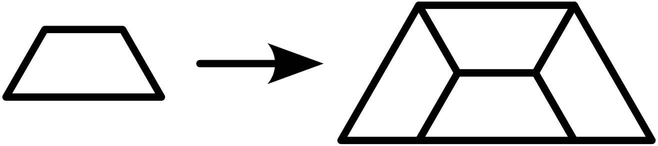
Geometric inflation rules

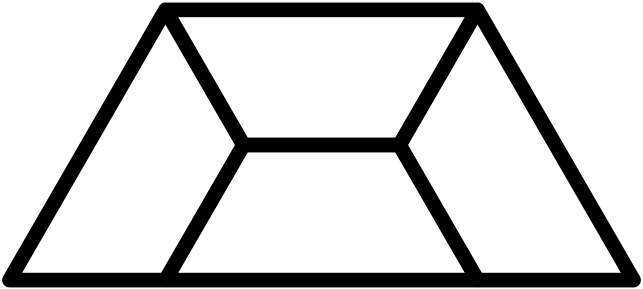
In the Fibonacci, associate as to an interval 'tile' of length $\varphi = \frac{1+\sqrt{5}}{2}$, and bs with tile of length 1.

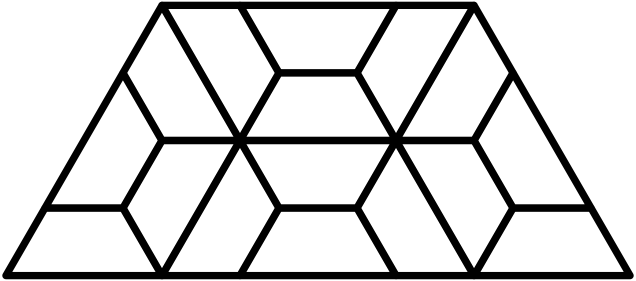
We thus define an associated geometric substitution rule, given by: first '**inflate** by φ', then **replace**:

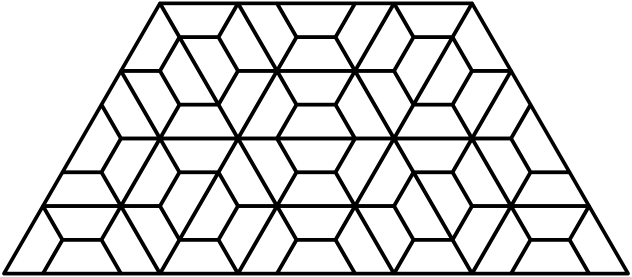


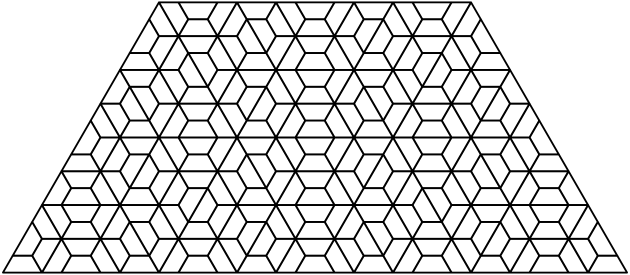
The ‘half-hex substitution’:

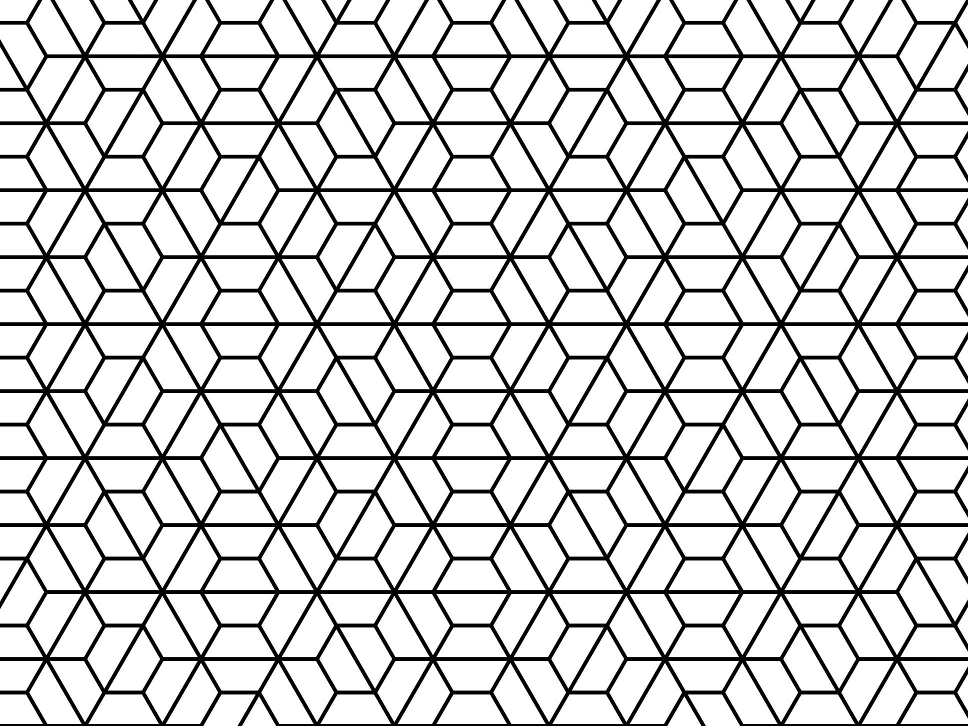


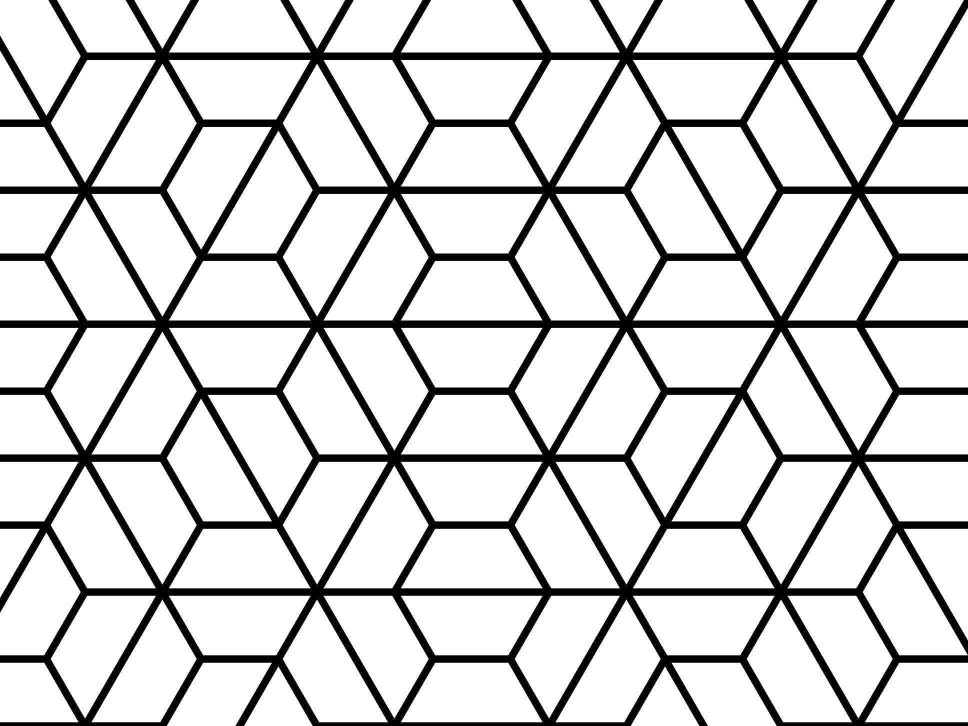












Recognisability

In the geometric and symbolic settings, recognisability is (a form of) injectivity of substitution σ, allowing unique desubstitution.

- Classical result of Mossé (1992/6): σ a primitive and non-periodic symbolic substitution $\implies$ σ is recognisable in X_σ ("unique desubstitution" within the subshift)

Recognisability

In the geometric and symbolic settings, recognisability is (a form of) injectivity of substitution σ, allowing unique desubstitution.

- Classical result of Mossé (1992/6): σ a primitive and non-periodic symbolic substitution $\implies$ σ is recognisable in X_σ ("unique desubstitution" within the subshift)
- Bezuglyi–Kwiatkowski–Medynets (2009): extended to non-primitive, non-periodic substitutions

Recognisability

In the geometric and symbolic settings, recognisability is (a form of) injectivity of substitution σ, allowing unique desubstitution.

- Classical result of Mossé (1992/6): σ a primitive and non-periodic symbolic substitution $\implies$ σ is recognisable in X_σ ("unique desubstitution" within the subshift)
- Bezuglyi–Kwiatkowski–Medynets (2009): extended to non-primitive, non-periodic substitutions
- Berthé–Steiner–Thuswaldner–Yassawi (2019): further extensions e.g., to results on 'full recognisability' and 'recognisable for aperiodic points' (subshifts can have mix of periodic and non-periodic) and S-adic shifts (sequences of substitutions)

Recognisability

In the geometric and symbolic settings, recognisability is (a form of) injectivity of substitution σ, allowing unique desubstitution.

- Classical result of Mossé (1992/6): σ a primitive and non-periodic symbolic substitution $\implies$ σ is recognisable in X_σ ("unique desubstitution" within the subshift)
- Bezuglyi–Kwiatkowski–Medynets (2009): extended to non-primitive, non-periodic substitutions
- Berthé–Steiner–Thuswaldner–Yassawi (2019): further extensions e.g., to results on 'full recognisability' and 'recognisable for aperiodic points' (subshifts can have mix of periodic and non-periodic) and S-adic shifts (sequences of substitutions)
- Béal–Perrin–Restivo (2023): any substitution (even one with erasing letters) is recognisable in X_σ for aperiodic points.
- B(é)PRS (2025): eventual recognisability and representability for aperiodic points in S-adic shifts with finite alphabet rank

In the geometric setting (arbitrary dimensions), the best-known result is Solomyak's, also known as 'unique composition':

Theorem (Sol98)

An FLC geometric **stone inflation** σ *that is* **primitive** *and aperiodic satisfies unique composition: for every tiling* $T \in \Omega_\sigma$*, there exists a unique* $T' \in \Omega_\sigma$ *with* $\sigma(T') = T$*.*

In the geometric setting (arbitrary dimensions), the best-known result is Solomyak's, also known as 'unique composition':

Theorem (Sol98)

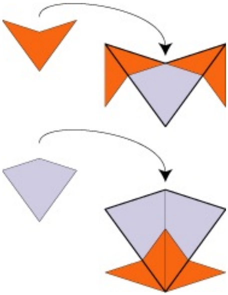
An FLC geometric **stone inflation** σ *that is* **primitive** *and aperiodic satisfies unique composition: for every tiling* $T \in \Omega_\sigma$*, there exists a unique* $T' \in \Omega_\sigma$ *with* $\sigma(T') = T$*.*

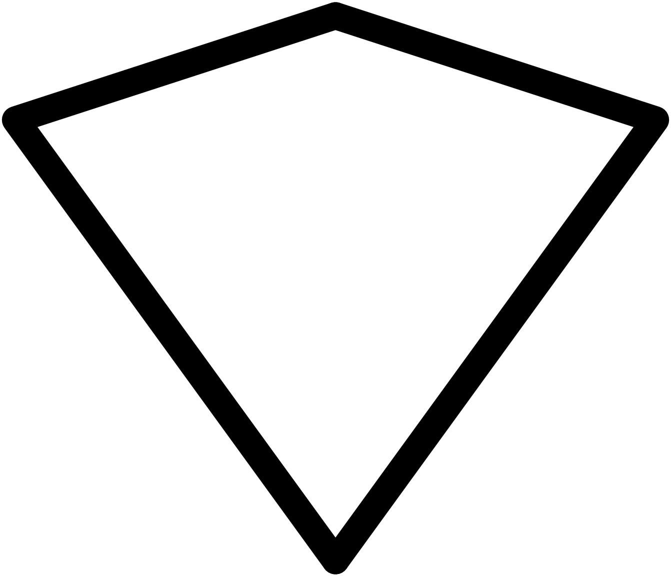
Original paper also gives the number of pre-images under σ when there are non-trivial translational periods.

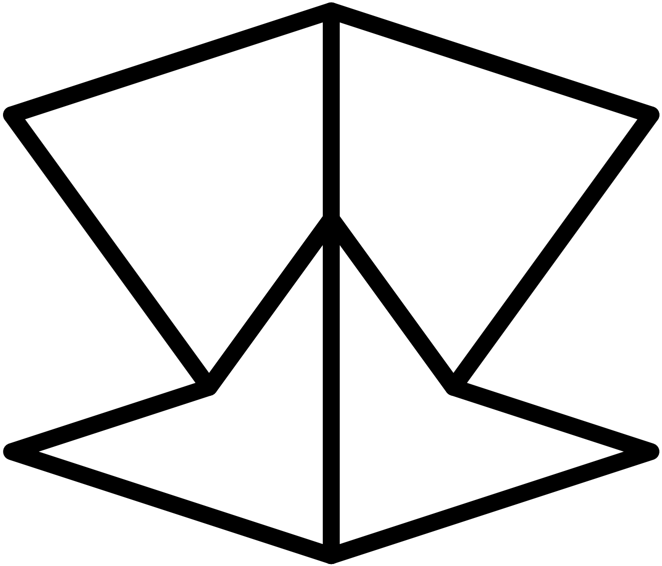
Some later results in non-primitive (non-minimal) case by Cortez–Solomyak (2011) e.g., 'admissible' stone inflation σ is injective $\iff$ σ non-periodic (i.e., Ω_σ consists of only non-periodic points). Some partial results when there are a mix of periodic and non-periodic elements.

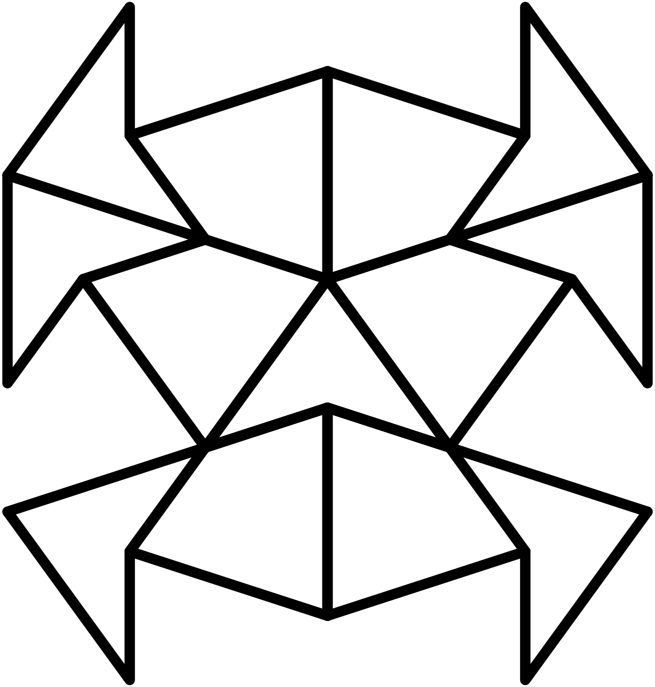
Other types of geometric substitutions

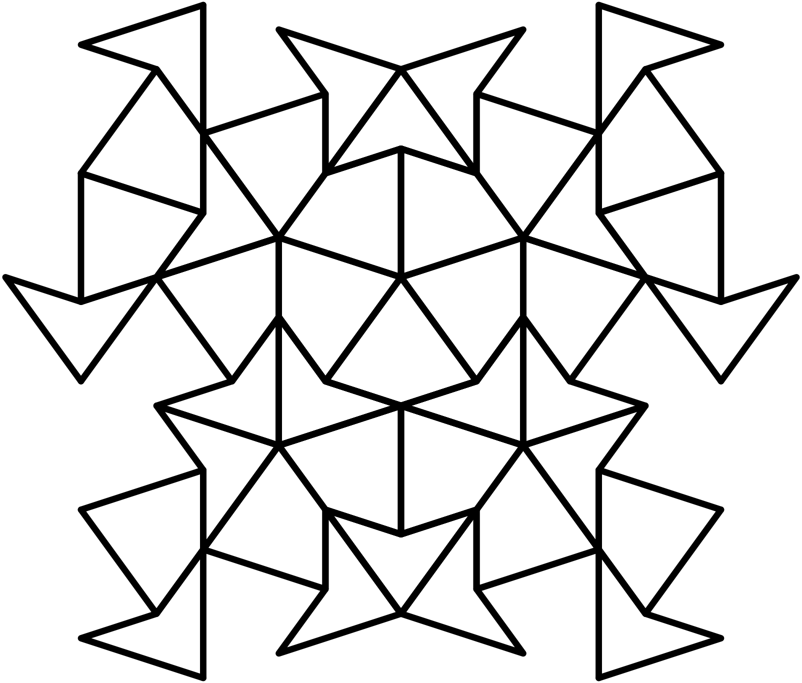
Many well-known tiling substitutions are not ‘stone inflations’ e.g., the Penrose kite and dart substitution:

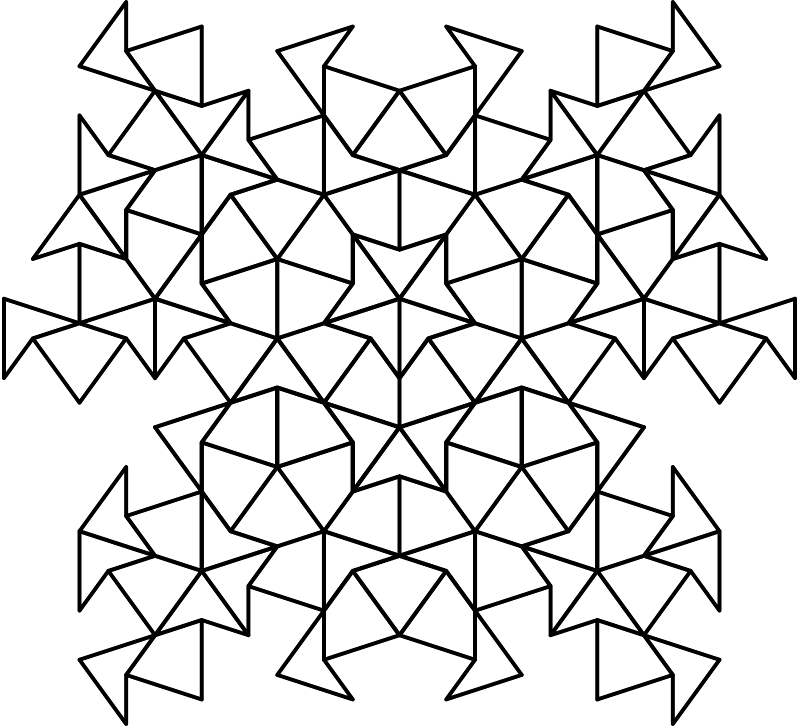


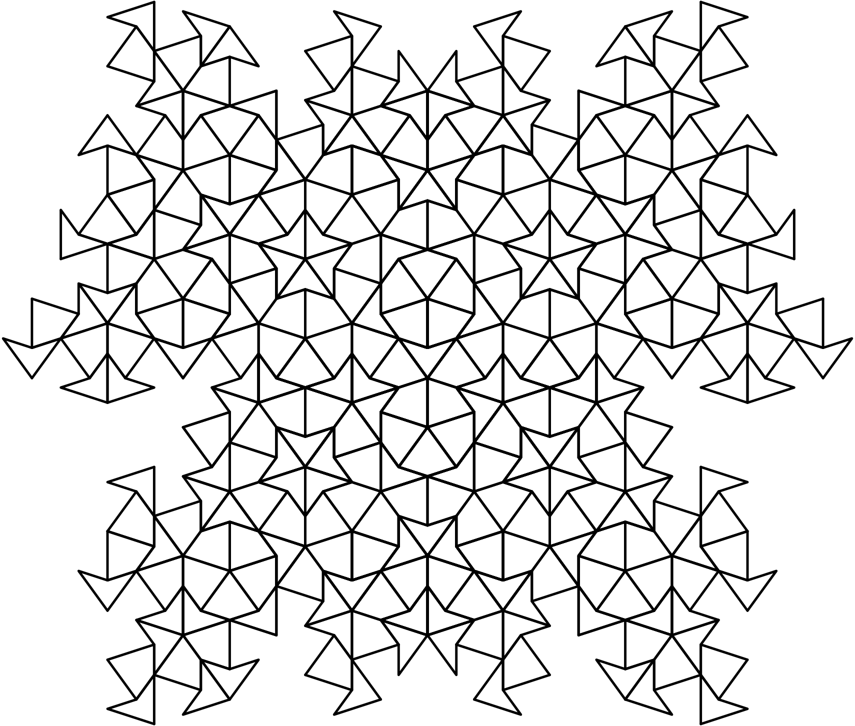


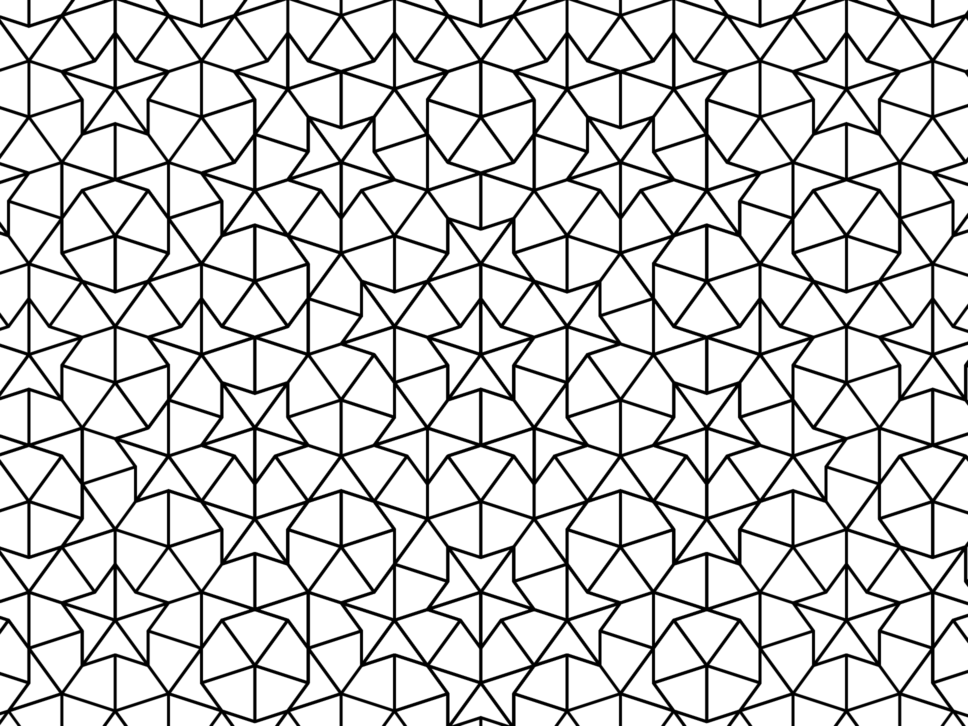


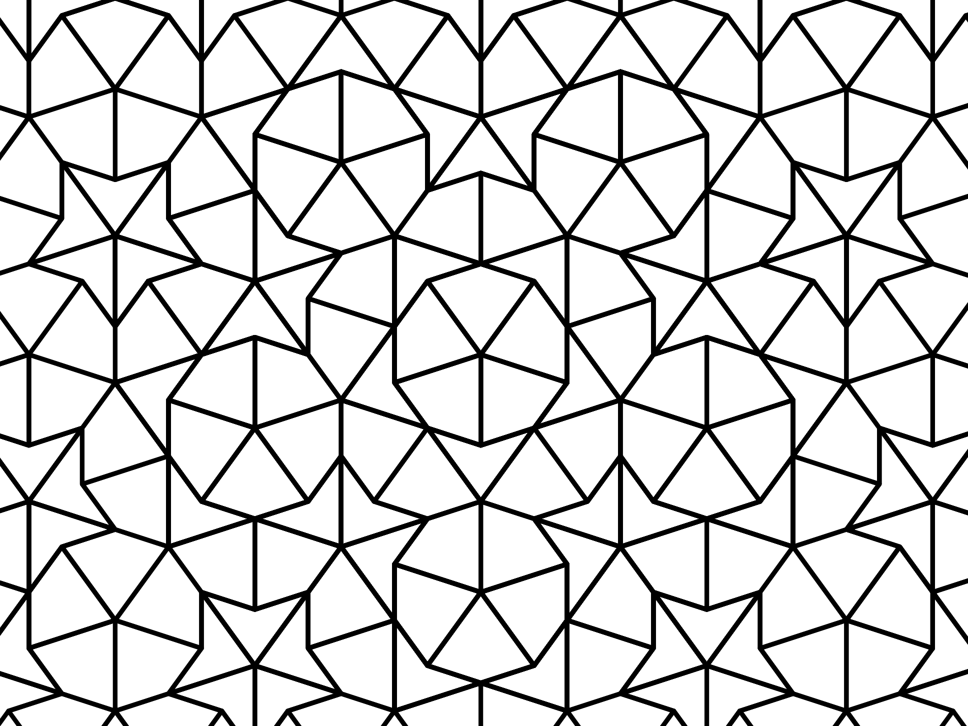










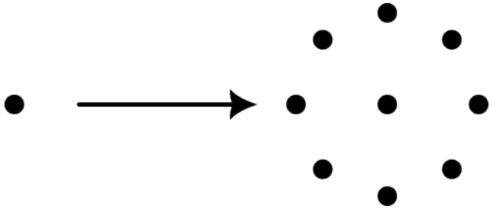


And what if we want to consider other kinds of ‘patterns’ (not just tilings)?

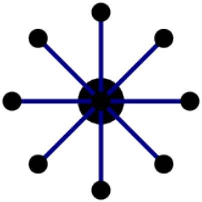
For instance, we could consider (coloured) point set substitutions.

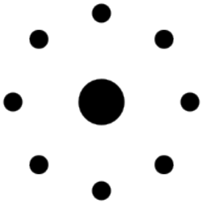
Example

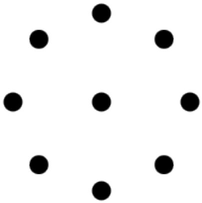
Inflate by $1 + \sqrt{2}$ then

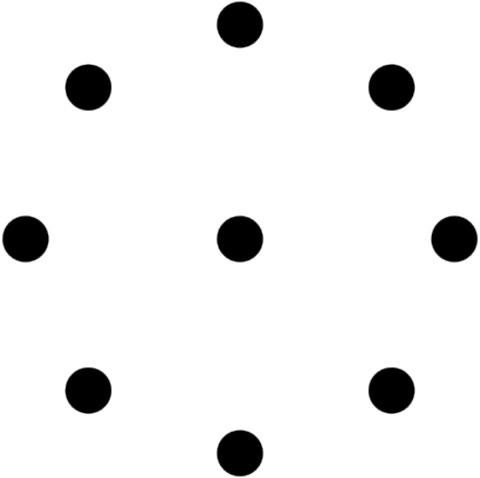


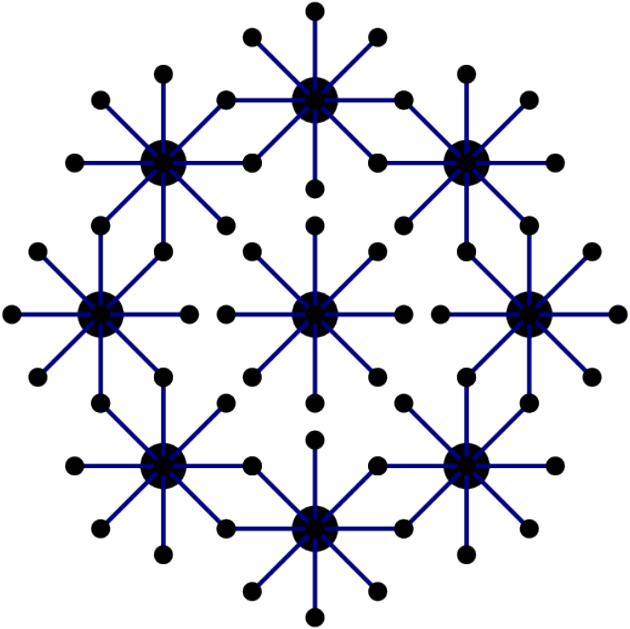
... and repeat...

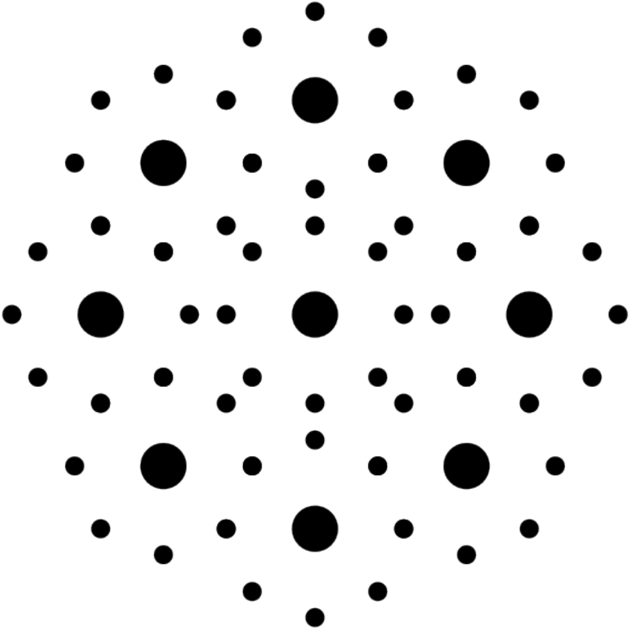


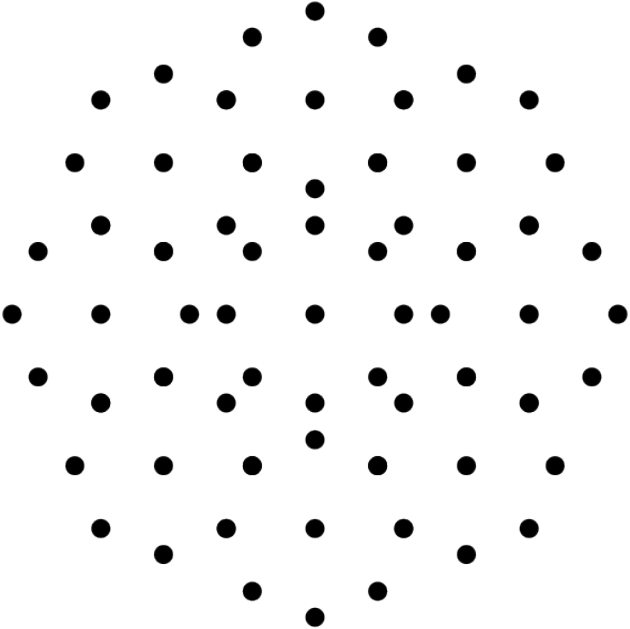


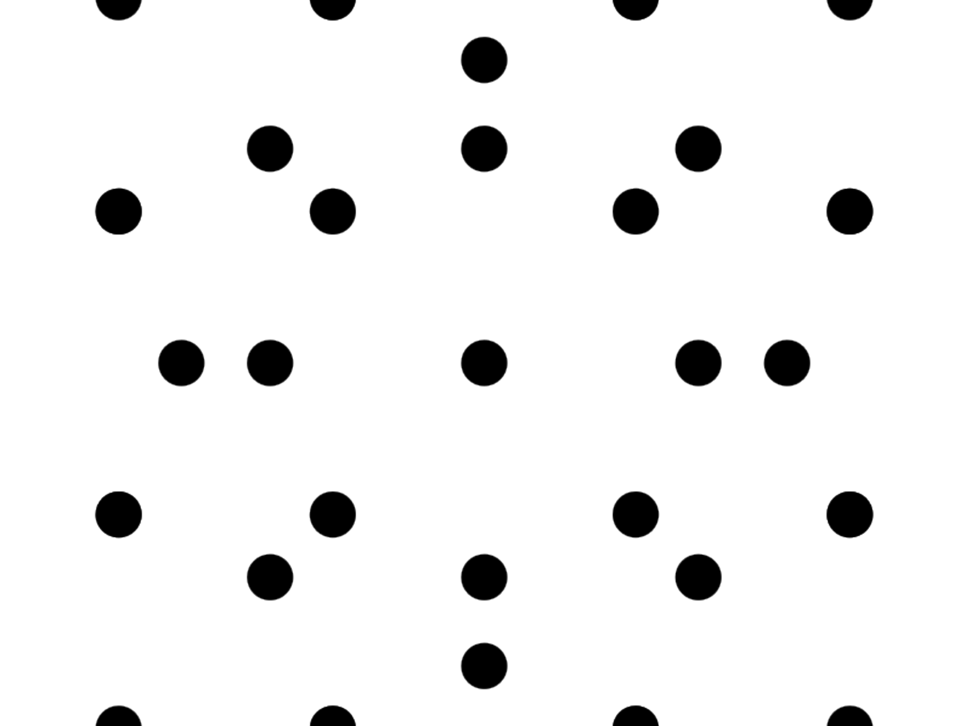


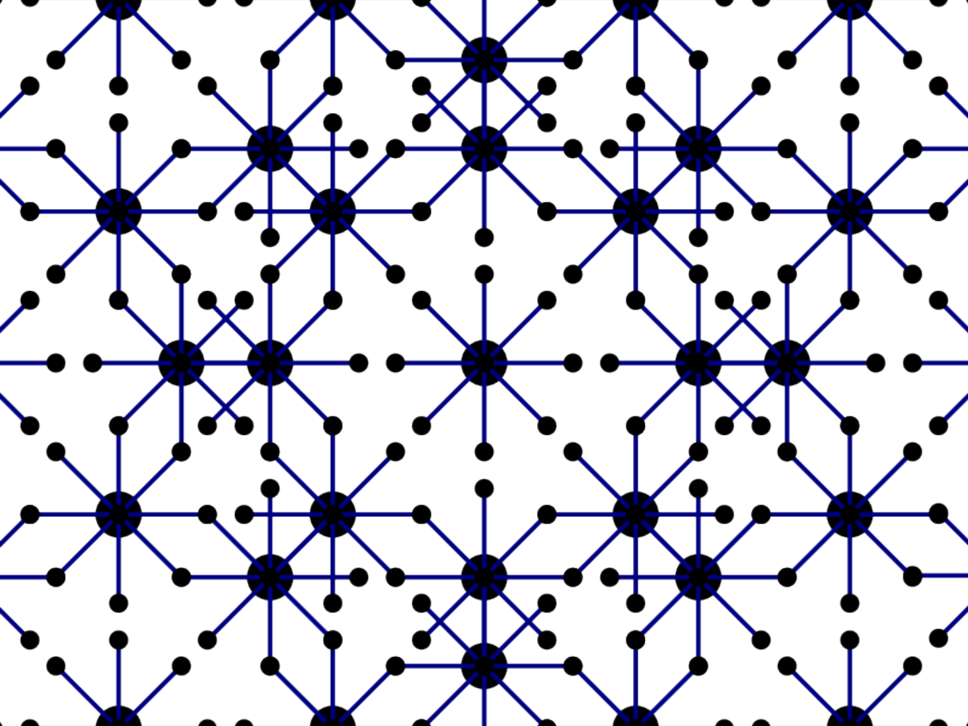


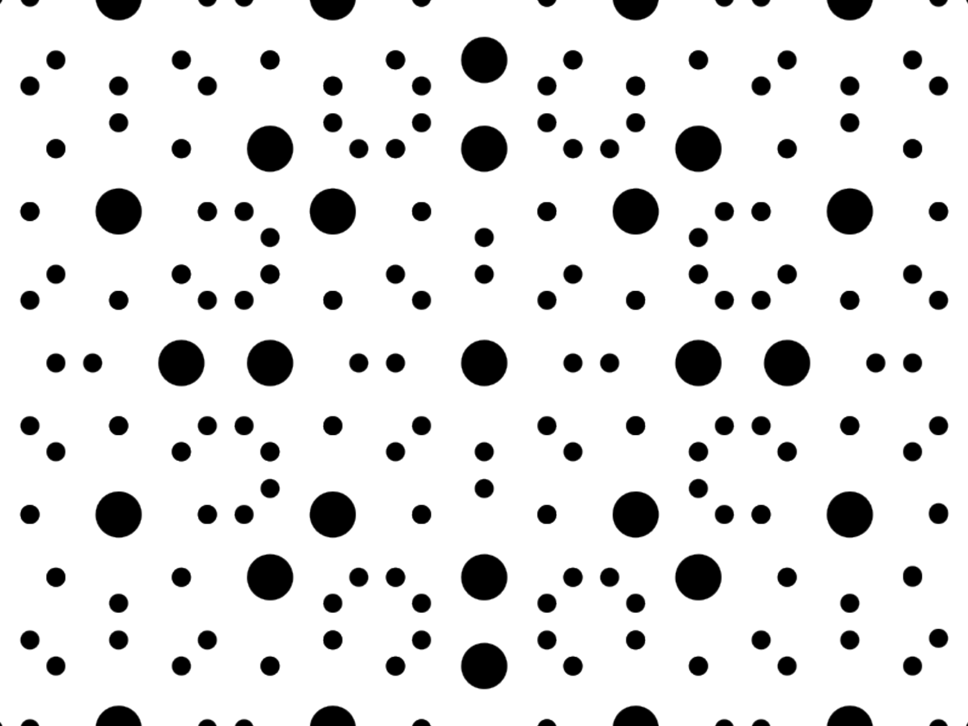


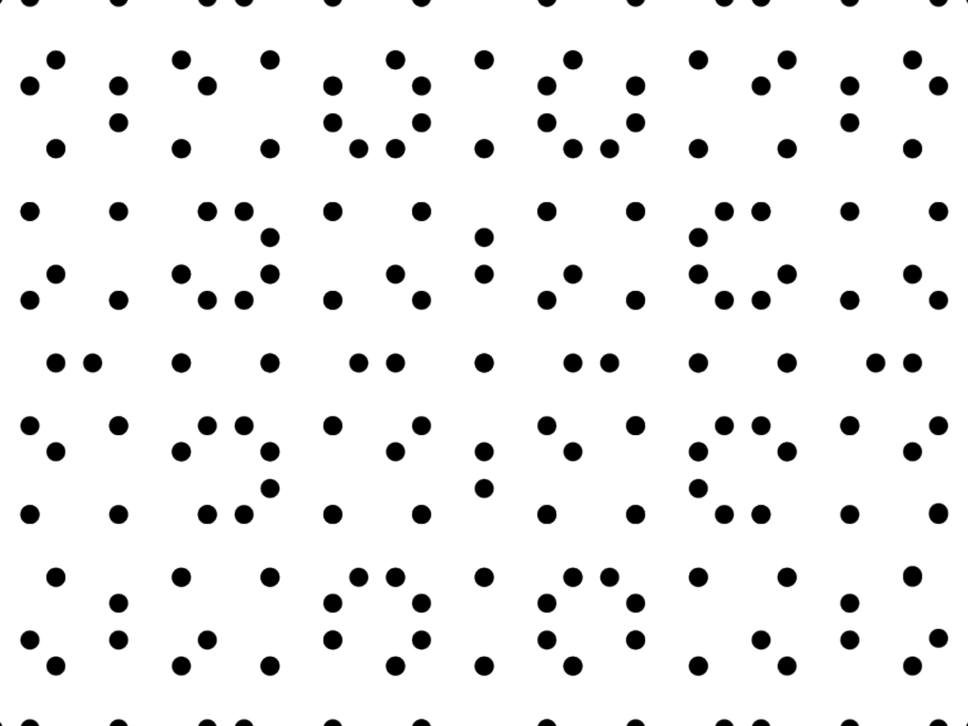


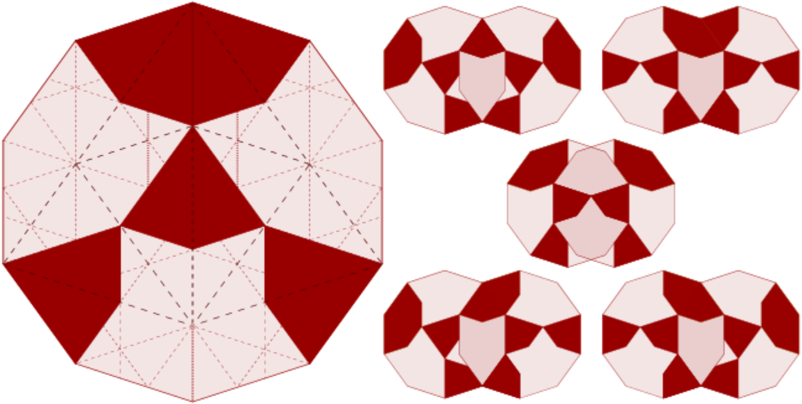


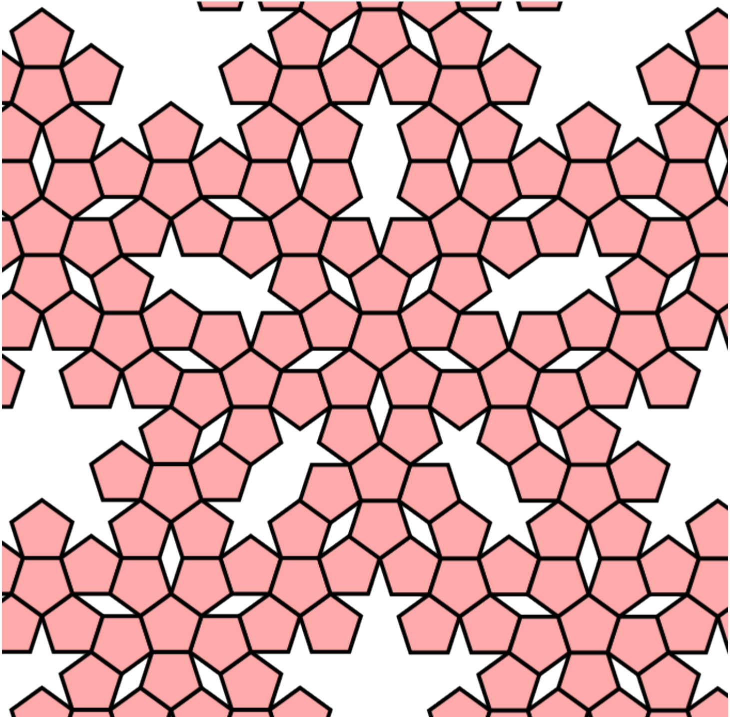












Recognisability shopping list

Some further things we could want for geometric recognisability:

- an **extended notion of** a Euclidean **pattern**: tilings, Delone sets, point sets with arbitrarily large 'holes', coverings with overlapping tiles, tilings with gaps, ...
- ... together with a natural notion of **when a space of patterns Ω is 'substitutional'** (not just stone inflations etc. and defining in terms of Ω, not generating 'from atoms')
- ... with a formula for $\#\sigma^{-1}(\mathcal{P})$ for all $\mathcal{P} \in \Omega$ (we may have a mix of periodic and non-periodic patterns in Ω).

I'll next try to explain the main result of [Wal25], that gives these. More precise definitions will be filled in after!

Recognisability shopping list

Some further things we could want for geometric recognisability:

- an **extended notion of** a Euclidean **pattern**: tilings, Delone sets, point sets with arbitrarily large 'holes', coverings with overlapping tiles, tilings with gaps, ...
- ... together with a natural notion of **when a space of patterns Ω is 'substitutional'** (not just stone inflations etc. and defining in terms of Ω, not generating 'from atoms')
- ... with a formula for $\#\sigma^{-1}(\mathcal{P})$ for all $\mathcal{P} \in \Omega$ (we may have a mix of periodic and non-periodic patterns in Ω).

I'll next try to explain the main result of [Wal25], that gives these. More precise definitions will be filled in after!

In the future, it would also be interesting to address:

- beyond FLC (e.g., 'pinwheel type' examples, or compact alphabets?)
- geometric S-adic systems?

Generalised recognisability (handwavey prerequisites)

Fix an ambient Euclidean space $E \cong \mathbb{R}^d$ of our patterns. We consider spaces Ω of patterns in E, with the **local topology**: patterns are close when they agree to a large radius about the origin, up to a small translation.

Assumption: Ω is compact (FLC) and Hausdorff with this topology. It is a dynamical system, with action of E by translation.

Generalised recognisability (handwavey prerequisites)

Fix an ambient Euclidean space $E \cong \mathbb{R}^d$ of our patterns. We consider spaces Ω of patterns in E, with the **local topology**: patterns are close when they agree to a large radius about the origin, up to a small translation.

Assumption: Ω is compact (FLC) and Hausdorff with this topology. It is a dynamical system, with action of E by translation.

Notation: $\mathcal{K}_{\mathcal{P}} =$ group of periods of the pattern $\mathcal{P}$ i.e., $\mathcal{K}_{\mathcal{P}} = \{x \in E \mid \mathcal{P} + x = \mathcal{P}\}$.

Let $L \colon E \to E$ be a linear expansion. We will introduce a notion of Ω being 'substitutional' with respect to L, called L-**sub**. This means we have a continuous map

$$\sigma = (S \circ L) \colon \Omega \to \Omega, \quad S \colon L\Omega \xrightarrow{\mathrm{LD}} \Omega;$$

the factor map S can be thought of as the 'subdivision' (or replacement rule) step. We demand it is **surjective**!

Recognisability for FLC L-sub pattern spaces

Theorem (Wal26)

$\forall \mathcal{P} \in \Omega$ and $\mathcal{P}' \in \sigma^{-1}(\mathcal{P})$, we have $\#\sigma^{-1}(\mathcal{P}) = [\mathcal{K}_{\mathcal{P}} : L\mathcal{K}_{\mathcal{P}'}]$.

Recognisability for FLC L-sub pattern spaces

Theorem (Wal26)

$\forall \mathcal{P} \in \Omega$ and $\mathcal{P}' \in \sigma^{-1}(\mathcal{P})$, we have $\#\sigma^{-1}(\mathcal{P}) = [\mathcal{K}_{\mathcal{P}} : L\mathcal{K}_{\mathcal{P}'}]$.

Lemma

There are only finitely many LI-classes ('languages') of patterns in Ω, and substitution permutes them.

Recognisability for FLC L-sub pattern spaces

Theorem (Wal26)

$\forall \mathcal{P} \in \Omega$ and $\mathcal{P}' \in \sigma^{-1}(\mathcal{P})$, we have $\#\sigma^{-1}(\mathcal{P}) = [\mathcal{K}_{\mathcal{P}} : L\mathcal{K}_{\mathcal{P}'}]$.

Lemma

There are only finitely many LI-classes ('languages') of patterns in Ω, and substitution permutes them.

Corollary

For a suitable power $n \in \mathbb{N}$ of σ (one fixing LI-classes) we have:

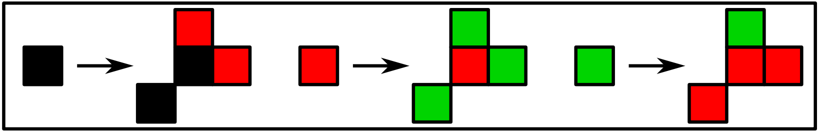
$$\forall \mathcal{P} \in \Omega,\ \#\sigma^{-n}(\mathcal{P}) = [\mathcal{K}_{\mathcal{P}} : L^n\mathcal{K}_{\mathcal{P}}].$$

Thus, non-periodic points have unique pre-images under (any power of) σ. Conversely, for a 'return discrete' pattern space (e.g., a space of tilings or Delone sets), periodic points have multiple pre-images under σ^n.

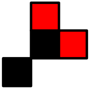
Example: ‘digit’ (or ‘constant shape’) substitutions on $\mathbb{Z}^d$

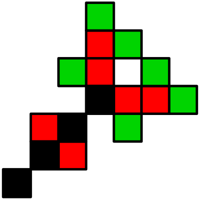
Let $E = \mathbb{R}^d$ and $L(x) = Mx$ expansive, M an integer $d \times d$ matrix.

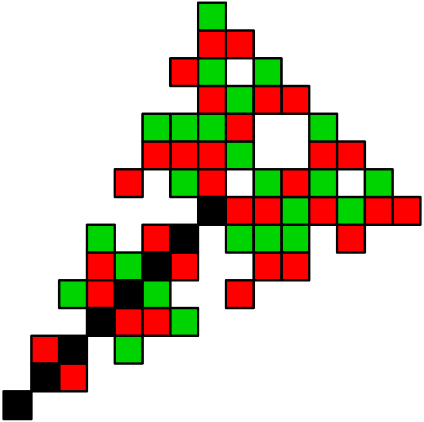
Take any fundamental domain $F \ni \mathbf{0}$ of $L(\mathbb{Z}^d) < \mathbb{Z}^d$. For a finite alphabet A, a **digit substitution** is a map $S\colon A \to A^F$.
For example, with $d = 2$, $L(x) = 2x$, $\#A = 3$,
$F = \{\mathbf{0}, e_1, e_2, -(e_1 + e_2)\}$,

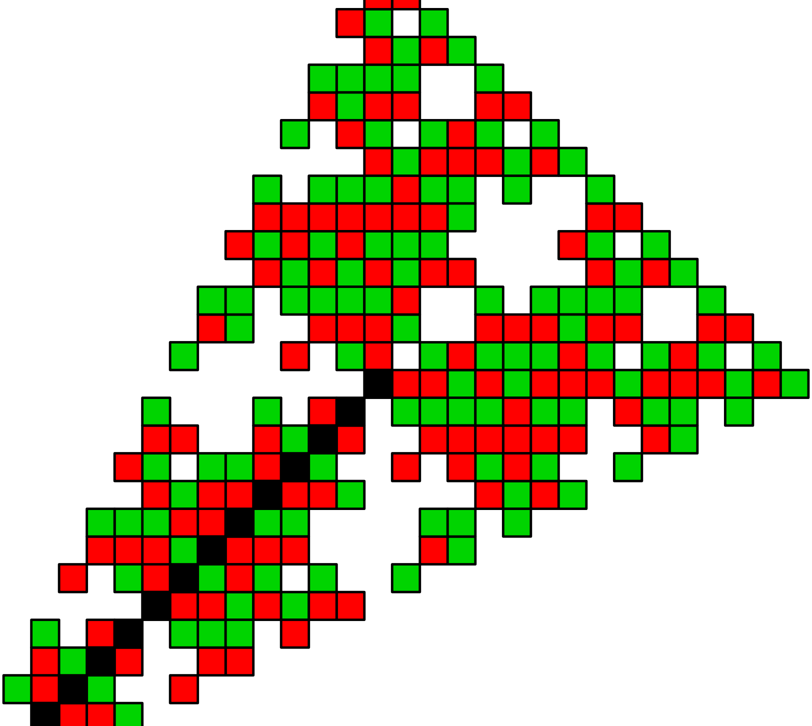


Can define compact, Hausdorff L-sub pattern spaces Ω_σ (with elements tilings of coloured hypercubes). S defines the map $S\colon L\Omega \xrightarrow{\mathrm{LD}} \Omega$ in obvious way. Surjectivity of σ (as we require) can be shown with a Cantor diagonalisation argument.









Non-tiling example: hull of 'single-dot pattern'

The result applies to some 'non-tiling' examples too:

Let $\mathcal{P}$ = point pattern of $E = \mathbb{R}^d$ of a single 'dot' at the origin, and $\emptyset$ the 'empty' pattern. We have the 'hull' (orbit closure) of $\mathcal{P}$:

$$\Omega = (\mathcal{P} + \mathbb{R}^d) \cup \{\emptyset\} \cong \mathbb{R}^d \cup \{\infty\} = S^d, \text{ the } d\text{-sphere.}$$

Ω is an L-sub pattern space (for any inflation L), where $S = \mathrm{Id}$: the 'dot' just substitutes to the same dot. Dynamically, $\sigma\colon \Omega \to \Omega$ has $\sigma(\mathcal{P}) = \mathcal{P}$ and, $\forall \mathcal{Q} \in \Omega \setminus \{\mathcal{P}\}$, $\sigma^n(\mathcal{Q}) \to \emptyset$ as $n \to \infty$.

σ is injective... even though Ω **contains the periodic point** $\emptyset$! Here, $\sigma^{-1}(\emptyset) = \{\emptyset\}$. This doesn't contradict the result: $\mathcal{K}_\emptyset = E$ and we have:

$$\#\sigma^{-1}(\emptyset) = [\mathcal{K}_\emptyset : L\mathcal{K}_\emptyset] = [E : E] = [E : E] = 1.$$

Formal definitions

Remainder of talk: I'll try to give more precise definitions for all the objects required for main results e.g., what's meant by a 'pattern'. Recall that $E \cong \mathbb{R}^d$ is our ambient (**E**uclidean) space.

Definition (Generalised pattern)

A **pattern** $\mathcal{P}$ is a function $\mathcal{P} \colon E \to A$. We denote $\mathcal{P}[x] := \mathcal{P}(x)$.

Analogy: a bi-infinite word is a map $w \colon \mathbb{Z} \to A$ i.e., $w \in A^{\mathbb{Z}}$. We consider 'colourings' of E instead of $\mathbb{Z}$ i.e., $\mathcal{P} \in A^E$.

Formal definitions

Remainder of talk: I'll try to give more precise definitions for all the objects required for main results e.g., what's meant by a 'pattern'. Recall that $E \cong \mathbb{R}^d$ is our ambient (**E**uclidean) space.

Definition (Generalised pattern)

A **pattern** $\mathcal{P}$ is a function $\mathcal{P} \colon E \to A$. We denote $\mathcal{P}[x] := \mathcal{P}(x)$.

Analogy: a bi-infinite word is a map $w \colon \mathbb{Z} \to A$ i.e., $w \in A^{\mathbb{Z}}$. We consider 'colourings' of E instead of $\mathbb{Z}$ i.e., $\mathcal{P} \in A^E$.

Easily covers examples including tilings and point sets (and more). Of course, more needed to ensure resulting objects aren't too wild. But all we'll need is our pattern spaces later being compact Hausdorff with the 'local topology'.

Note: no requirement that A is finite. A perfectly reasonable (and FLC) pattern can have $\#A = \infty$ (and this can easily change between 'MLD' patterns).

Transformed patterns, and patches

Definition (Transformation of a pattern)

For $\mathcal{P} \in A^E$ and linear automorphism $T\colon E \to E$ (e.g., a translation), $T\mathcal{P}$ defined in the obvious way:

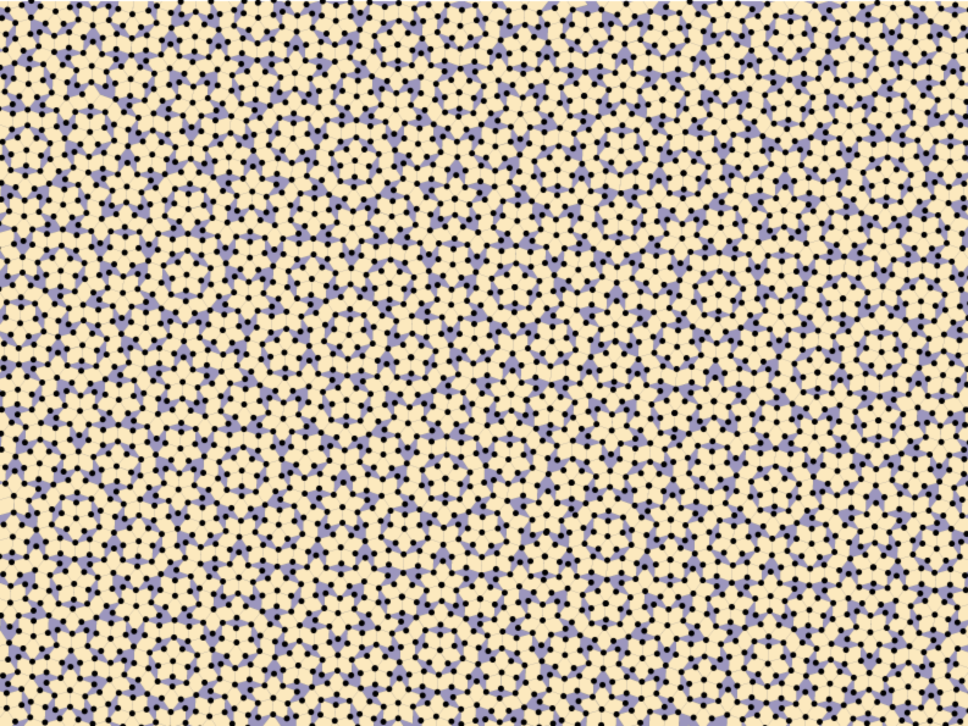
$$(T\mathcal{P})[x] \coloneqq \mathcal{P}[T^{-1}x] \text{ i.e., } (T\mathcal{P})[Tx] = \mathcal{P}[x].$$

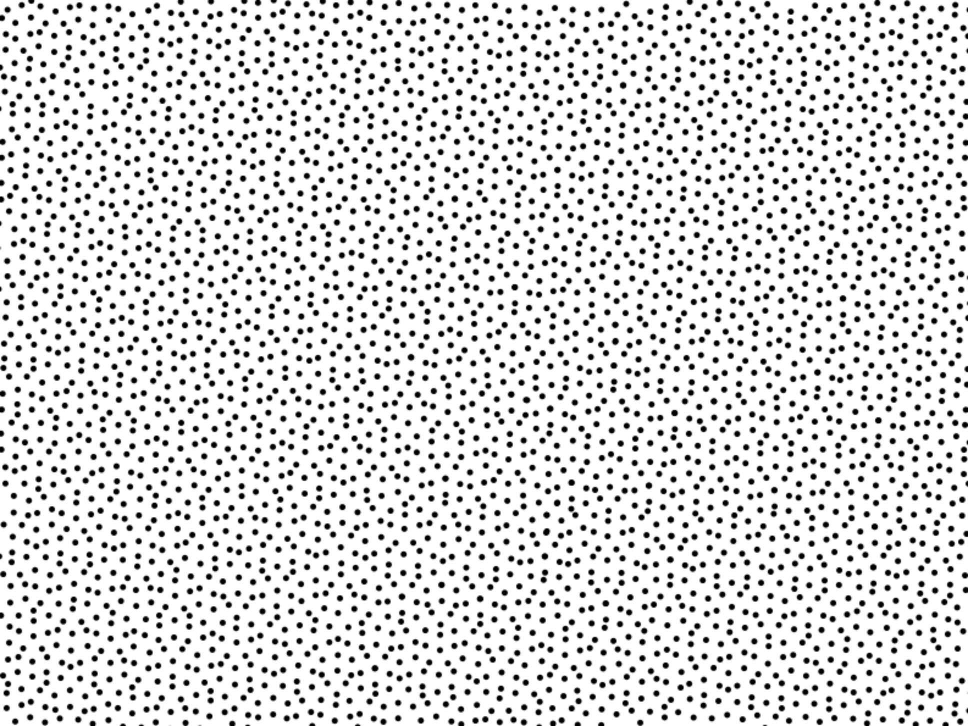
The analogue of a 'factor' of a bi-infinite word is a patch:

Definition (Patch of a pattern)

For $x \in E$ and $U \subset E$ we define $\mathcal{P}[x, U] \in A^U$ by $u \mapsto \mathcal{P}[x + u]$. We call $\mathcal{P}[x, U]$ a **patch** if U is compact. For $r \geq 0$, we define r-**patches** $\mathcal{P}[x, r] \coloneqq \mathcal{P}[x, B_r]$.

Thus, $U =$ the 'shape' of the patch and $x =$ 'choice of centre'. If, say, $\mathcal{P}[x, U] = \mathcal{P}[y, U]$, then we think: *the patches of shape U centred at x and y are the same* (modulo translation from x to y).





(Mutual) local derivation

Definition (LD and MLD)

$\mathcal{Q}$ is **locally derivable (LD) from** $\mathcal{P}$ ($\mathcal{P} \overset{\mathrm{LD}}{\rightsquigarrow} \mathcal{Q}$) if, for some $c \geq 0$,

$$\mathcal{P}[x, c] = \mathcal{P}[y, c] \implies \mathcal{Q}[x] = \mathcal{Q}[y].$$

If $\mathcal{Q}$ is locally derivable from $\mathcal{P}$ and vice versa ($\mathcal{P} \overset{\mathrm{MLD}}{\leftrightsquigarrow} \mathcal{Q}$), we call $\mathcal{P}$ and $\mathcal{Q}$ **mutually locally derivable** (**MLD**).

LD is a geometric analogue of a 'sliding block code'.
MLD formalises 'the same, up to reversible and locally-defined redecoration' (a reversible 'recoding').

Example

Take a tiling of 2×1 rectangles (meeting vertex-to-vertex), defining a pattern $\mathcal{P}$. Cut all rectangles in half to define the pattern $\mathcal{Q}$ (periodic tiling of squares). The 'cutting' is locally defined: $\mathcal{P} \overset{\mathrm{LD}}{\rightsquigarrow} \mathcal{Q}$. But we do **not** have $\mathcal{Q} \overset{\mathrm{LD}}{\rightsquigarrow} \mathcal{P}$. 'Joining' squares is not locally-defined (which pairs do we join?).

Local indistinguishability

Definition (Local indistinguishability/isomorphism)

$\mathcal{P}$ is **locally indistinguishable from** $\mathcal{Q}$ ($\mathcal{P} \overset{\mathrm{LI}}{\sqsubset} \mathcal{Q}$) if every patch of $\mathcal{P}$ appears in $\mathcal{Q}$ i.e., $\forall x \in E$ and compact $U \subset E$, $\exists y \in E$ with $\mathcal{P}[x, U] = \mathcal{Q}[y, U]$.

If $\mathcal{P}$ is locally indistinguishable from $\mathcal{Q}$ and vice versa ($\mathcal{P} \overset{\mathrm{LI}}{\simeq} \mathcal{Q}$), we call $\mathcal{P}$ and $\mathcal{Q}$ **locally isomorphic**.

Symbolic Dynamics analogue: $\mathcal{P} \overset{\mathrm{LI}}{\sqsubset} \mathcal{Q}$ when language of $\mathcal{P}$ is a subset of that of $\mathcal{Q}$ $\iff$ orbit closure of $\mathcal{P}$ contained in that of $\mathcal{Q}$.

Example

$\forall \mathcal{P} \in A^E$ and $x \in E$, $\mathcal{P} \overset{\mathrm{LI}}{\simeq} \mathcal{P} + x$.

Example

Define 1d tilings of labelled interval tiles with associated patterns $\mathcal{P}$ and $\mathcal{Q}$, resp. from the symbolic sequences $\cdots aaaaaaa \cdots$ and $\cdots aaabaaa \cdots$. Then $\mathcal{P} \overset{\mathrm{LI}}{\sqsubset} \mathcal{Q}$, but the reverse is false.

Pattern spaces

Definition (Pattern spaces)

Let $\Omega \subseteq A^E$ (some *set* of patterns). Take any pattern $\mathcal{P} \in A^E$ (not necessarily in Ω).

Write $\mathcal{P} \overset{\mathrm{LI}}{\in} \Omega$ if, for all $x \in E$ and compact $U \subset E$, there exists $\mathcal{Q} \in \Omega$, $y \in E$ with $\mathcal{P}[x, U] = \mathcal{Q}[y, U]$. That is:
'*all patches of $\mathcal{P}$ appear in some patterns in Ω*'.

We call Ω a **pattern space** if $\mathcal{P} \overset{\mathrm{LI}}{\in} \Omega \implies \mathcal{P} \in \Omega$.

Analogy: a non-empty $X \subseteq A^{\mathbb{Z}}$ is a subshift $\iff \forall w \in A^{\mathbb{Z}}$ with the language of w contained in the language of X, $w \in X$.

Example

For a pattern $\mathcal{P}$, we define $\Omega_{\mathcal{P}} := \{\mathcal{Q} \in A^E \mid \mathcal{Q} \overset{\mathrm{LI}}{\sqsubset} \mathcal{P}\}$, the **(translational) hull** of $\mathcal{P}$. It is a pattern space, the 'orbit closure' of $\mathcal{P}$ ($\Omega_{\mathcal{P}} = \overline{\mathcal{P} + E}$).

Topology of pattern spaces

We give pattern spaces the 'local topology':

(Which is most naturally defined in terms of a **uniformity**; we also have a notion of sequences being Cauchy etc.)

$\mathcal{P}$, $\mathcal{Q} \in \Omega$ 'close' if $\mathcal{P}[x, r] = \mathcal{Q}[y, r]$ for x, $y \in E$ small and r large (small translates agree to a large radius around origin).

Symbolic Dynamics analogue: Ω is like the suspension of a subshift (origin can lie in middles of tiles, not just on boundaries etc.).

Topology of pattern spaces

We give pattern spaces the 'local topology':

(Which is most naturally defined in terms of a **uniformity**; we also have a notion of sequences being Cauchy etc.)

$\mathcal{P}$, $\mathcal{Q} \in \Omega$ 'close' if $\mathcal{P}[x,r] = \mathcal{Q}[y,r]$ for x, $y \in E$ small and r large (small translates agree to a large radius around origin).

Symbolic Dynamics analogue: Ω is like the suspension of a subshift (origin can lie in middles of tiles, not just on boundaries etc.).

Dynamical system: $E \cong \mathbb{R}^d$ acts continuously on Ω (by translation) i.e., $x \cdot \mathcal{P} := \mathcal{P} + x$.

Ω is Hausdorff given some very mild restrictions ('well separated').

Ω is always complete (comes from $\mathcal{P} \overset{\mathrm{LI}}{\in} \Omega \implies \mathcal{P} \in \Omega$). Compact when Ω satisfies 'finite local complexity' (FLC). This can be defined in terms of patterns. For tilings: **only finitely many tile types, meeting in only finitely many different ways**.

Local derivation maps between pattern spaces

Definition

$f\colon \Omega \to \Omega'$ is an **LD map** (written $f\colon \Omega \xrightarrow{\text{LD}} \Omega'$) if $\exists c \geq 0$ s.t. $\forall\ \mathcal{P}$, $\mathcal{Q} \in \Omega$, x, $y \in E$,

$$\mathcal{P}[x,c] = \mathcal{Q}[y,c] \implies (f\mathcal{P})[x] = (f\mathcal{Q})[y].$$

Again, geometric analogue of **sliding block code factor map**. However, note: in geometric setting, this is **stronger** than just being a factor map (see Clark–Sadun shape changes etc.).

Can prove (amongst lots more) that LD maps are always continuous and commute with translation, and $\mathcal{P} \overset{\text{LD}}{\rightsquigarrow} \mathcal{Q}$ induces a canonical LD map $\Omega_{\mathcal{P}} \xrightarrow{\text{LD}} \Omega_{\mathcal{Q}}$.

Local derivation maps between pattern spaces

Definition

$f\colon \Omega \to \Omega'$ is an **LD map** (written $f\colon \Omega \xrightarrow{\mathrm{LD}} \Omega'$) if $\exists c \geq 0$ s.t. $\forall$ $\mathcal{P}$, $\mathcal{Q} \in \Omega$, x, $y \in E$,

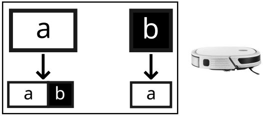
$$\mathcal{P}[x,c] = \mathcal{Q}[y,c] \implies (f\mathcal{P})[x] = (f\mathcal{Q})[y].$$

Again, geometric analogue of **sliding block code factor map**. However, note: in geometric setting, this is **stronger** than just being a factor map (see Clark–Sadun shape changes etc.).

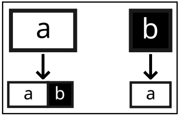
Can prove (amongst lots more) that LD maps are always continuous and commute with translation, and $\mathcal{P} \overset{\mathrm{LD}}{\rightsquigarrow} \mathcal{Q}$ induces a canonical LD map $\Omega_{\mathcal{P}} \xrightarrow{\mathrm{LD}} \Omega_{\mathcal{Q}}$.

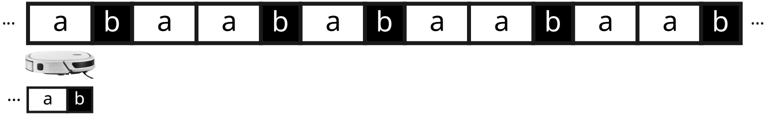
We now use this concept to give a general notion of a pattern (space) being ‘substitutional’.

Local derivations are analogues of 'sliding block codes':

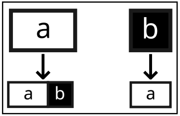


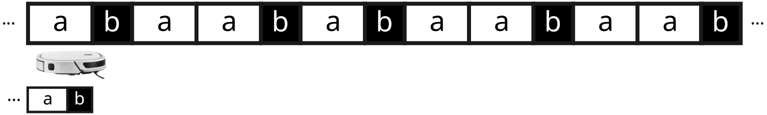
Local derivations are analogues of 'sliding block codes':



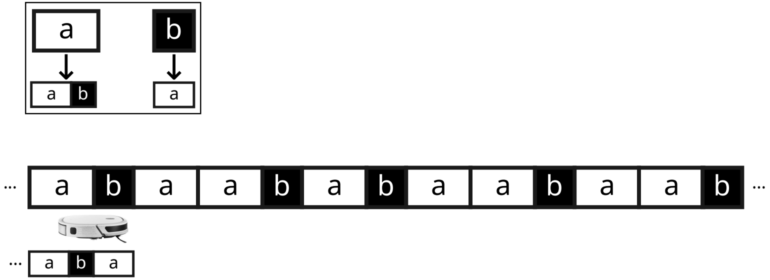


Local derivations are analogues of 'sliding block codes':

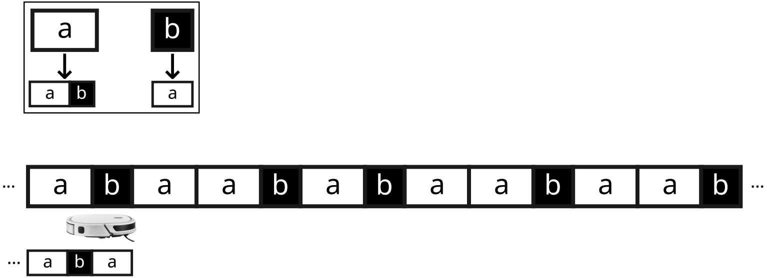




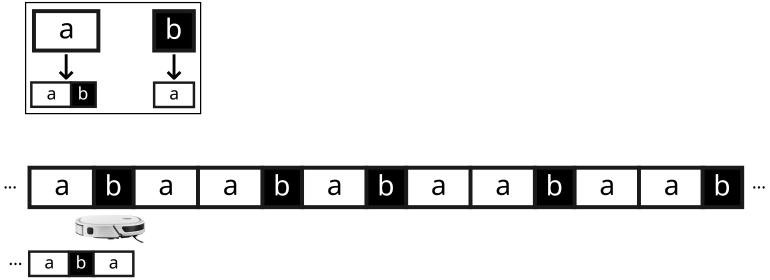
Local derivations are analogues of ‘sliding block codes’:



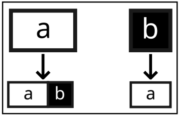
Local derivations are analogues of ‘sliding block codes’:

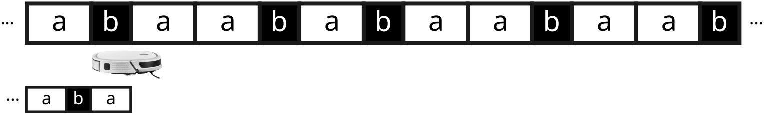


Local derivations are analogues of ‘sliding block codes’:

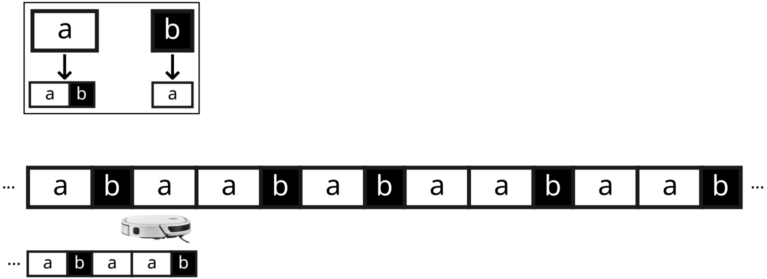


Local derivations are analogues of 'sliding block codes':

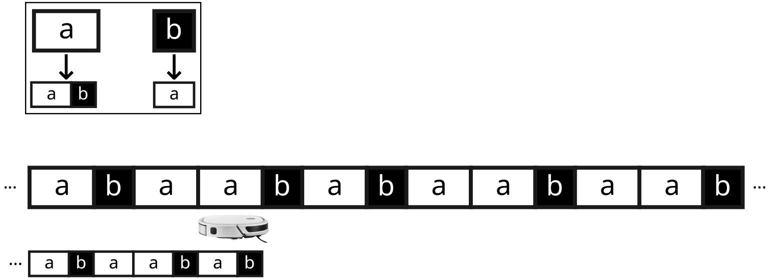




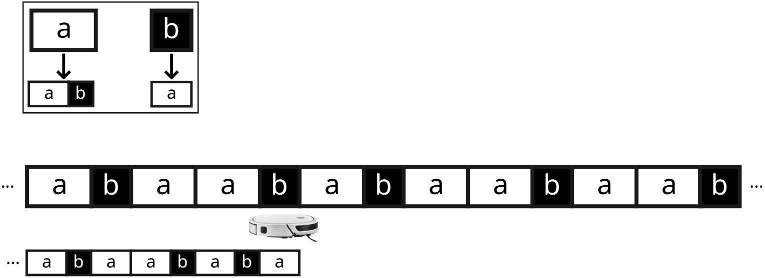
Local derivations are analogues of ‘sliding block codes’:



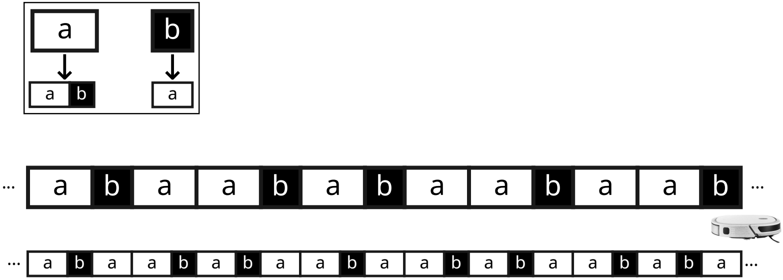
Local derivations are analogues of ‘sliding block codes’:

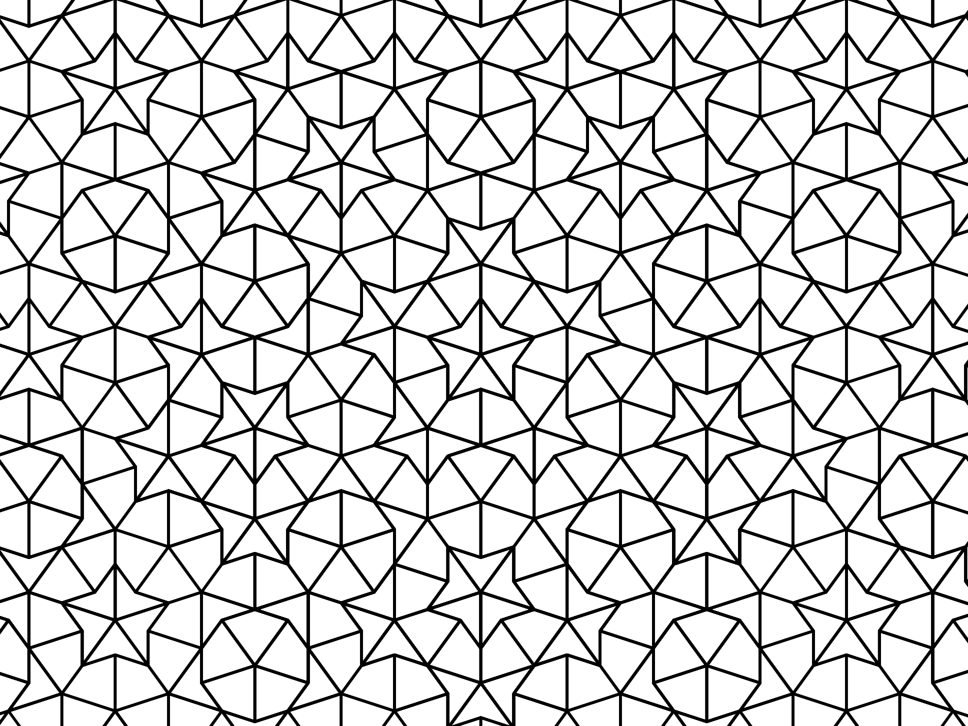


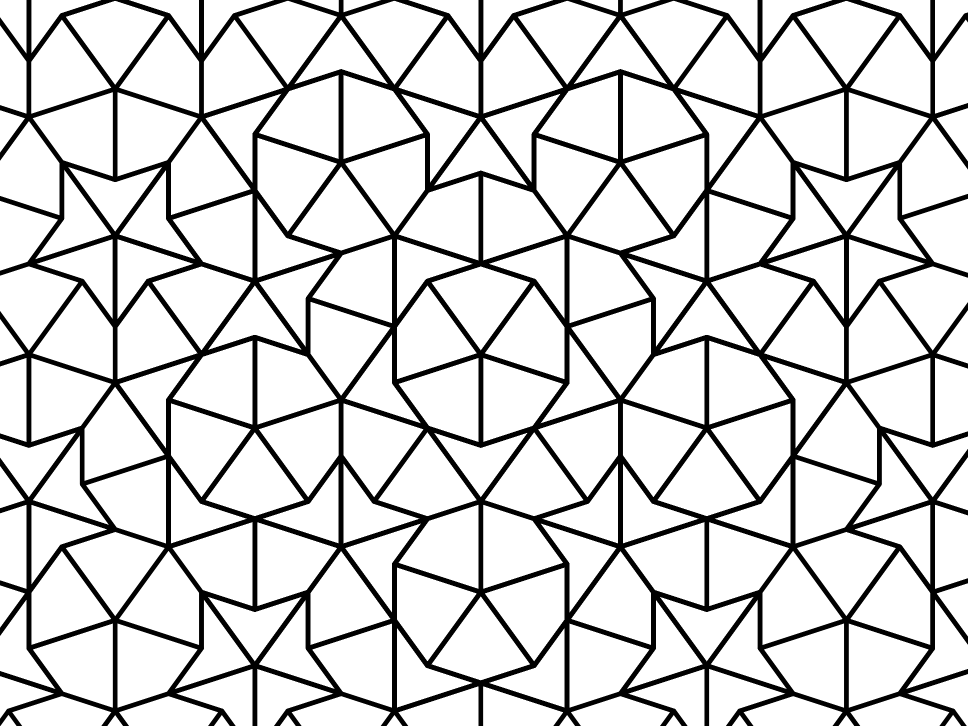
Local derivations are analogues of 'sliding block codes':

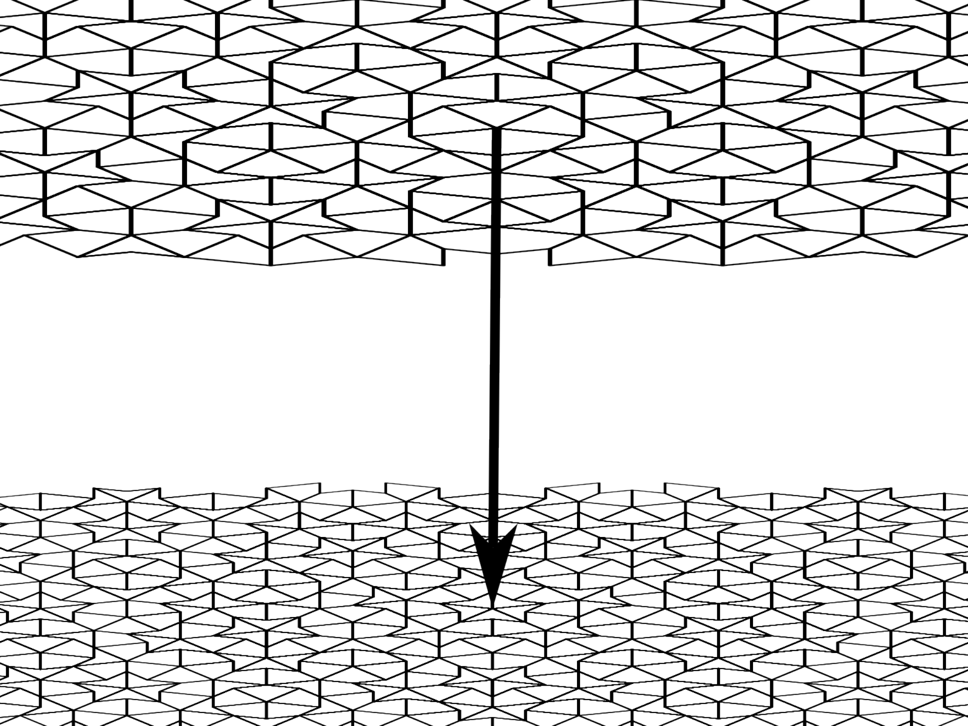


Local derivations are analogues of ‘sliding block codes’:









L-sub pattern spaces

We let $L\colon E \to E$ be an **expansive linear map**. For a pattern space Ω of patterns in E, we define $L\Omega = \{L\mathcal{P} \mid \mathcal{P} \in \Omega\}$, the space of inflated patterns (which is itself a pattern space).

Definition (L-sub pattern space)

Ω is called L-**sub** if there exists a surjective (!) LD map

$$S\colon L\Omega \xrightarrow{\mathrm{LD}} \Omega,$$

which we call **subdivision**. We then define the **substitution map** by $\sigma := S \circ L\colon \Omega \to \Omega$.

This definition says: *every pattern is the 'subdivision' of some super-pattern (the inflation of another in Ω), by a local rule*.

Individual L-sub patterns

Definition (L-sub for a single pattern)

An individual pattern $\mathcal{P}$ is L-**sub** if $\exists$ some pattern $\mathcal{P}' \overset{\mathrm{LI}}{\sqsubset} \mathcal{P}$ (that is, $\mathcal{P}' \in \Omega_{\mathcal{P}}$), called a **predecessor**, with $L\mathcal{P}' \overset{\mathrm{LD}}{\rightsquigarrow} \mathcal{P}$.

Can show that if $\mathcal{P}$ is L-sub (with pred. $\mathcal{P}'$) and $\Omega_{\mathcal{P}}$ cpt HD then $\mathcal{P} \overset{\mathrm{LI}}{\simeq} \mathcal{P}'$ and induced map of $L\mathcal{P}' \overset{\mathrm{LD}}{\rightsquigarrow} \mathcal{P}$ defines a surjection

$$S \colon L\Omega_{\mathcal{P}} \xrightarrow{\mathrm{LD}} \Omega_{\mathcal{P}}.$$

Thus, a cpt HD hull (orbit closure) $\Omega_{\mathcal{P}}$ of an L-sub pattern is itself naturally L-sub, so our main theorems apply to such cases.

Main recognisability results (reminder)

Theorem (Wal25)

If $\Omega \subseteq A^E$ *is a compact, Hausdorff pattern space,* $L\colon E \to E$ *is expansive and* Ω *is* L*-sub (meaning: we have a surjective* $S\colon L\Omega \xrightarrow{\mathrm{LD}} \Omega$*, giving* $\sigma := (S \circ L)\colon \Omega \to \Omega$*), then:*

$\forall \mathcal{P} \in \Omega$, $\mathcal{P}' \in \sigma^{-1}(\mathcal{P})$,

$$\#\sigma^{-1}(\mathcal{P}) = [\mathcal{K}_{\mathcal{P}} : L\mathcal{K}_{\mathcal{P}'}],$$

where $\mathcal{K}_{\mathcal{Q}} =$ *the group of periods of a pattern* $\mathcal{Q}$*. All pre-images are translation-equivalent.*

Patterns that aren't 'discretely periodic' (i.e., with $\mathcal{K}_{\mathcal{P}}$ *connected) have unique pre-images under* σ *('recog. for aperiodic points').*

Conversely, $\exists n \in \mathbb{N}$ *such that discretely periodic points have multiple pre-images under* σ^n*.*

Summary/final thoughts

- convenient framework for general FLC geometric 'patterns' (tilings, point sets and more) and of spaces Ω of such patterns
- natural notion of Ω being substitutional: 'sliding block code from exduced system'. This nicely avoids complexities of building up pattern spaces from 'inflate, replace' rules on 'atoms' (of many potential types!).
  - Q: useful in symbolic setting? i.e., define notion of a *subshift* being substitutional, rather than defining subshifts defined by languages generated from substitutions?
- ... yet similar to usual notions ('inflate, replace', pseudo-self affine etc.)
- gives very well-behaved notion of being 'substitutional': nice answers when used for question 'When is a cut and project set substitutional?'
- No primitivity (minimality) required for recognisability results, and we have a formula for $\#\sigma^{-1}(\mathcal{P})$ for all $\mathcal{P} \in \Omega$ in terms of groups of periods.
- Q: Can we drop FLC? There are likely extensions of the result to some classes of non-FLC patterns. What about compact alphabet substitutions?
- Q: Can we prove (and even formulate!) an S-adic version?

Summary/final thoughts

- convenient framework for general FLC geometric 'patterns' (tilings, point sets and more) and of spaces Ω of such patterns
- natural notion of Ω being substitutional: 'sliding block code from exduced system'. This nicely avoids complexities of building up pattern spaces from 'inflate, replace' rules on 'atoms' (of many potential types!).
  - Q: useful in symbolic setting? i.e., define notion of a *subshift* being substitutional, rather than defining subshifts defined by languages generated from substitutions?
- ... yet similar to usual notions ('inflate, replace', pseudo-self affine etc.)
- gives very well-behaved notion of being 'substitutional': nice answers when used for question 'When is a cut and project set substitutional?'
- No primitivity (minimality) required for recognisability results, and we have a formula for $\#\sigma^{-1}(\mathcal{P})$ for all $\mathcal{P} \in \Omega$ in terms of groups of periods.
- Q: Can we drop FLC? There are likely extensions of the result to some classes of non-FLC patterns. What about compact alphabet substitutions?
- Q: Can we prove (and even formulate!) an S-adic version?

Thank you!

Cut and project sets